1998 Calculus BC Free-Response Questions

CALCULUS BC

Section II

Time—1 hour and 30 minutes

Number of problems—6

Percent of total grade—50

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS OR PARTS OF PROBLEMS ON THIS SECTION OF THE EXAMINATION.

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

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1. Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{\frac{3}{2}}$, the x–axis, and the y–axis.
   (a) Find the area of the region R.
   (b) Find the volume of the solid generated when R is revolved about the x–axis.
   (c) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.

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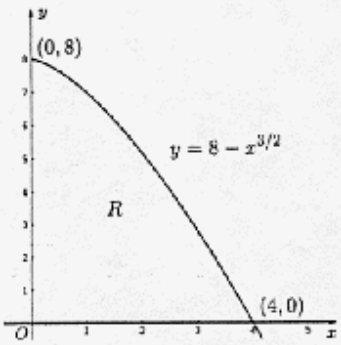
(a)



$$A = \int_0^4 \left(8 - x^{3/2}\right) dx$$

$$= 8x - \frac{2}{5}x^{5/2}\Big|_0^4 = 32 - \frac{64}{5} = \frac{96}{5} = 19.2$$

3 { 2: integral; 1: integrand; 1: limits; 1: answer }

(b) $$V = \pi \int_0^4 \left(8 - x^{3/2}\right)^2 dx$$

$$= \frac{576\pi}{5} = 115.2\pi \approx 361.911$$

3 { 2: integral; 1: integrand; 1: limits and constant; 1: answer }

(c) $$\pi \int_0^k \left(8 - x^{3/2}\right)^2 dx = \frac{115.2\pi}{2}$$

[or

$$\pi \int_0^k \left(8 - x^{3/2}\right)^2 dx = \pi \int_k^4 \left(8 - x^{3/2}\right)^2 dx$$

]

$$\int_0^k \left(8 - x^{3/2}\right)^2 dx = 57.6$$

$$\int_0^k (64 - 16x^{3/2} + x^3)\, dx = 57.6$$

$$64k - \frac{32}{5}k^{5/2} + \frac{k^4}{4} = 57.6$$

$k \approx 0.995$ or 0.994

3 { 1: integral with k in limits; 1: equates volumes; 1: answer }

Note: 0/1 for answer in each part if no setup points earned

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2. Let f be the function given by $f(x) = 2xe^{2x}$.
   (a) Find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.
   (b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.
   (c) What is the range of f?
   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

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   (c) What is the range of f?
   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) $\lim_{x\to-\infty} 2xe^{2x} = 0$

$\lim_{x\to\infty} 2xe^{2x} = \infty$ or DNE

2 { 1: 0 as $x \to -\infty$; 1: ∞ or DNE as $x \to \infty$

(b) $f'(x) = 2e^{2x} + 2x \cdot 2 \cdot e^{2x} = 2e^{2x}(1 + 2x) = 0$

if $x = -1/2$

$f(-1/2) = -1/e$ or -0.368 or -0.367

$-1/e$ is an absolute minimum value because:

(i) $f'(x) < 0$ for all $x < -1/2$ and
$f'(x) > 0$ for all $x > -1/2$

–or–

(ii)

x		$-1/2$	
$f'(x)$	$-$	0	$+$

and $x = -1/2$ is the only critical number

3 { 1: solves $f'(x) = 0$; 1: evaluates f at student's critical point; 0/1 if not local minimum from student's derivative ; 1: justifies absolute minimum value; 0/1 for a local argument; 0/1 without explicit symbolic derivative

Note: 0/3 if no absolute minimum based on student's derivative

(c) Range of $f = [-1/e, \infty)$
or $[-0.367, \infty)$
or $[-0.368, \infty)$

1: answer

Note: must include the left–hand endpoint; exclude the right–hand "endpoint"

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1 + bx) = 0$

if $x = -1/b$

At $x = -1/b$, $y = -1/e$

y has an absolute minimum value of $-1/e$ for all nonzero b

3 { 1: sets $y' = be^{bx}(1 + bx) = 0$; 1: solves student's $y' = 0$; 1: evaluates y at a critical number and gets a value independent of b

Note: 0/3 if only considering specific values of b

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1998 AP Calculus AB Scoring Guidelines

AB 2 Board Note # 1

Part (d)

3/3 Argument with the following three ingredients:

1. The graph of $y = bxe^{bx}$ is a horizontal compression or expansion (with a reflection across the y–axis if $b < 0$) of the graph of $y = xe^x$.
2. The range of $y = bxe^{bx}$ is therefore the same as the range of $y = xe^x$.
3. Therefore the absolute minimum value of $y = bxe^{bx}$ is the same for all (non–zero) values of b.

0/3 Analyzing the horizontal compression/expansion of graphs of $y = bxe^{bx}$ for specific values of b.

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3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.
   (a) Write the third–degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.
   (b) Write the fourth–degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.
   (c) Write the third–degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.
   (d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

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1998 Calculus BC Scoring Guidelines

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   (d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $P_3(f)(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$

$f(0.2) \approx P_3(f)(0.2) =$

$$5 - 3(0.2) + \frac{0.04}{2} + \frac{2(0.008)}{3} =$$

4.425

3
- 2: $5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$
  <−1> each incorrect term, extra term, or $+\cdots$
- 1: approximates $f(0.2)$

<−1> for incorrect use of $=$

(b) $P_4(g)(x) = P_2(f)(x^2) = 5 - 3x^2 + \frac{1}{2}x^4$

2: $P_2(f)(x^2)$
<−1> each incorrect or extra term

(c) $P_3(h)(x) = \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$

$$= \left[5t - \frac{3}{2}t^2 + \frac{1}{6}t^3\right]_0^x$$

$$= 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$$

2
- 1: $P_3(h)(x) = \int_0^x P_2(f)(t)\,dt$
- 1: answer
  0/1 if any incorrect or extra terms

(d) $h(1) = \int_0^1 f(t)\,dt$

cannot be determined because $f(t)$ is known only for $t = 0$ and $t = 1$

2
- 1: $h(1)$ cannot be determined
- 1: reason

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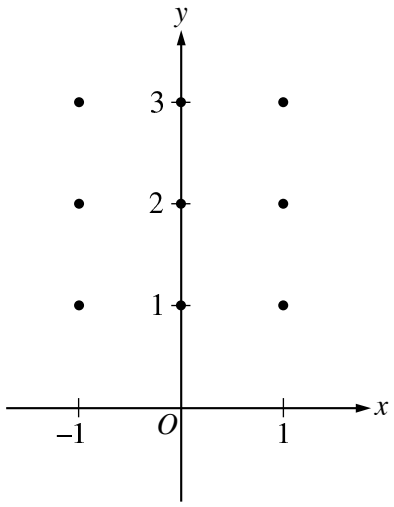
4. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

   (a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.



   (b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

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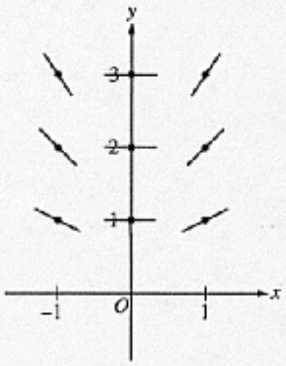
(a)



1: line segments at nine points with negative - zero - positive slope left to right and increasing steepness bottom to top at $x = 1$ and $x = -1$

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 3 + \frac{1}{2}(0)(3)(0.1) = 3$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$= 3 + \frac{1}{2}(0.1)(3)(0.1)$

$= 3 + \frac{.03}{2} = 3.015$

2
- 1: Euler's Method equations or equivalent table
- 1: answer (not eligible without first point)

Special Case: 1/2 for first iteration 3.015 and second iteration 3.045

(c) $\frac{dy}{dx} = \frac{xy}{2}$

$\int \frac{dy}{y} = \int \frac{x}{2}\,dx$

$\ln|y| = \frac{1}{4}x^2 + C_1$

$y = Ce^{x^2/4}$

$3 = Ce^0 \implies C = 3$

$y = 3e^{x^2/4}$

$f(0.2) = 3e^{.04/4} = 3e^{.01} = 3.030$

6
- 1: separates variables
- 1: antiderivative of dy term
- 1: antiderivative of dx term
- 1: solves for y
- 1: solves for constant of integration
- 1: evaluates $f(0.2)$

Note: max 4/6 [1-1-1-0-0-1] if no constant of integration

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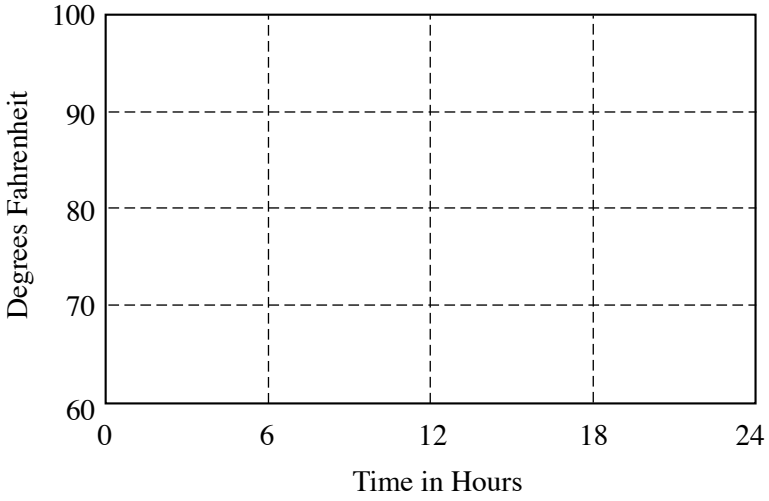
5. The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

(a) Sketch the graph of F on the grid below.



(b) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(c) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(d) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24–hour period?

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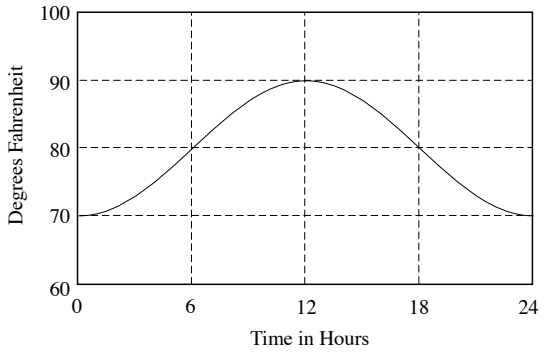

1: bell–shaped graph
minimum 70 at $t = 0$, $t = 24$ only
maximum 90 at $t = 12$ only

(b) $$\text{Avg.} = \frac{1}{14-6}\int_6^{14}\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right]dt$$

$$= \frac{1}{8}(697.2957795)$$

$$= 87.162 \text{ or } 87.161$$

$$\approx 87^\circ \text{ F}$$

3
- 2: integral
  - 1: limits and $1/(14-6)$
  - 1: integrand
- 1: answer
- 0/1 if integral not of the form $\frac{1}{b-a}\int_a^b F(t)\,dt$

(c) $$\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78 \ge 0$$

$$2 - 10\cos\left(\frac{\pi t}{12}\right) \ge 0$$

$$\left.\begin{matrix}5.230\\ \text{or}\\ 5.231\end{matrix}\right\} \le t \le \left\{\begin{matrix}18.769\\ \text{or}\\ 18.770\end{matrix}\right.$$

2
- 1: inequality or equation
- 1: solutions with interval

(d) $$C = 0.05\int_{\substack{5.231\\ \text{or}\\ 5.230}}^{\substack{18.769\\ \text{or}\\ 18.770}}\left(\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78\right)dt$$

$$= 0.05(101.92741) = 5.096 \approx \$5.10$$

3
- 2: integral
  - 1: limits and 0.05
  - 1: integrand
- 1: answer
- 0/1 if integral not of the form $k\int_a^b (F(t) - 78)\,dt$

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6. A particle moves along the curve defined by the equation $y = x^3 - 3x$. The x–coordinate of the particle, $x(t)$, satisfies the equation $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{2t+1}}$, for $t \geq 0$ with initial condition $x(0) = -4$.
   (a) Find $x(t)$ in terms of t.
   (b) Find $\dfrac{dy}{dt}$ in terms of t.
   (c) Find the location and speed of the particle at time $t = 4$.

END OF EXAMINATION

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(a) $x(t) = \int \frac{1}{\sqrt{2t+1}}\,dt$

$x(t) = \sqrt{2t+1} + C$

$x(0) = -4 = 1 + C \implies C = -5$

$x(t) = \sqrt{2t+1} - 5$

3 { 1: $x(t) = \int \frac{dt}{\sqrt{2t+1}}$; 1: $x(t) = \sqrt{2t+1} + C$; 1: evaluates C

(b) $y = x^3 - 3x$

$\frac{dy}{dt} = 3x^2\frac{dx}{dt} - 3\frac{dx}{dt}$

$= (3x^2 - 3)\frac{dx}{dt}$

$= \left[3\left(\sqrt{2t+1} - 5\right)^2 - 3\right]\left[\frac{1}{\sqrt{2t+1}}\right]$

2: answer

<−1> each error

Note: failure to express $\frac{dy}{dt}$ solely in terms of t is a single error

(c) $x(4) = \sqrt{9} - 5 = -2$

$y(4) = (-2)^3 - 3(-2) = -2$

Location at $t = 4$ is $(-2, -2)$

$\left.\frac{dx}{dt}\right|_{t=4} = \frac{1}{3}$

$\left.\frac{dy}{dt}\right|_{t=4} = \frac{3(3-5)^2 - 3}{3} = 3$

Speed $= \sqrt{\left(\frac{1}{3}\right)^2 + 3^2} = \sqrt{\frac{82}{9}} = 3.018$

4 { 1: position ; 1: evaluates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at $t = 4$; 1: uses speed formula ; 1: answer

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1999

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CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

Number of problems—6

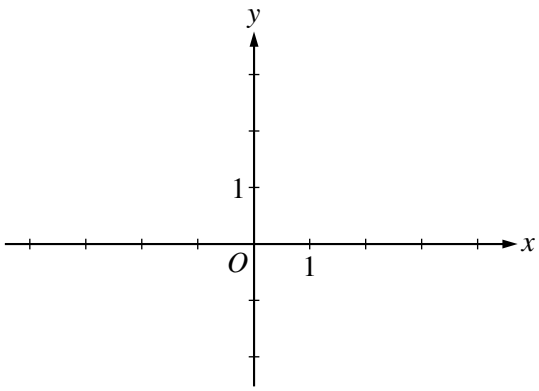
Percent of total grade—50

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

---

1. A particle moves in the xy-plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \frac{t^2}{2} - \ln(1 + t)$ and $y(t) = 3 \sin t$.

   (a) Sketch the path of the particle in the xy-plane below. Indicate the direction of motion along the path.



   (b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

   (c) At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

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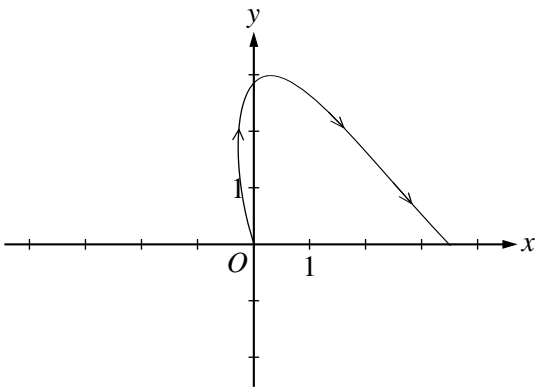
(a)



2 { 1: graph; 1: direction }

(b) $x'(t) = t - \dfrac{1}{1+t} = 0$

$t^2 + t - 1 = 0$

$t = \dfrac{-1+\sqrt{5}}{2}$ or $t = 0.618$ in $[0, \pi]$

$x(0.618) = -0.290 \qquad y(0.618) = 1.738$

3 { 1: $x'(t) = 0$; 1: solution for t; 1: position }

(c) $x(t) = \dfrac{t^2}{2} - \ln(1+t) = 0$

$t = 1.285$ or 1.286

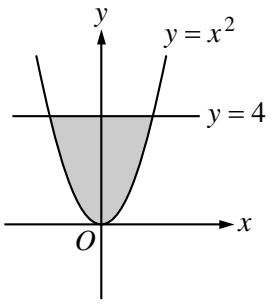
$x'(t) = t - \dfrac{1}{1+t} \qquad y'(t) = 3\cos t$

speed $= \sqrt{(x'(1.286))^2 + (y'(1.286))^2} = 1.196$

$x''(t) = 1 + \dfrac{1}{(1+t)^2} \qquad y''(t) = -3\sin t$

acceleration vector $= \langle x''(1.286), y''(1.286) \rangle$
$= \langle 1.191, -2.879 \rangle$

4 { 1: $x(t) = 0$; 1: solution for t; 1: speed; 1: acceleration vector }



2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated by revolving R about the x-axis.

   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

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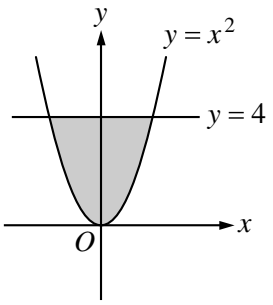

(a) Area $= \displaystyle\int_{-2}^{2} (4 - x^2)\,dx$

$= 2\displaystyle\int_{0}^{2} (4 - x^2)\,dx$

$= 2\left[4x - \dfrac{x^3}{3}\right]_0^2$

$= \dfrac{32}{3} = 10.666$ or 10.667

2 { 1: integral; 1: answer }

(b) Volume $= \pi\displaystyle\int_{-2}^{2} \left(4^2 - (x^2)^2\right) dx$

$= 2\pi\displaystyle\int_{0}^{2} (16 - x^4)\,dx$

$= 2\pi\left[16x - \dfrac{x^5}{5}\right]_0^2$

$= \dfrac{256\pi}{5} = 160.849$ or 160.850

3 { 1: limits and constant; 1: integrand; 1: answer }

(c) $\pi\displaystyle\int_{-2}^{2} \left[(k - x^2)^2 - (k - 4)^2\right] dx = \dfrac{256\pi}{5}$

4 { 1: limits and constant; 2: integrand <−1> each error; 1: equation }

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

   (a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Using correct units, explain the meaning of your answer in terms of water flow.

   (b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

   (c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$.
   Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period.
   Indicate units of measure.

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3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.

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24	9.6

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   (b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

   (c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

(a) $\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$
$= 6[10.4 + 11.2 + 11.3 + 10.2]$
$= 258.6$ gallons

This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.

3 { 1: $R(3) + R(9) + R(15) + R(21)$; 1: answer ; 1: explanation }

(b) Yes;

Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.

2 { 1: answer ; 1: MVT or equivalent }

(c) Average rate of flow
$\approx$ average value of $Q(t)$
$= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$
$= 10.785$ gal/hr or 10.784 gal/hr

3 { 1: limits and average value constant ; 1: $Q(t)$ as integrand ; 1: answer }

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

   (a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

   (b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

   (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

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   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$f(1.5) \approx T_3(f,2)(1.5)$

$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$

$= -4.958\overline{3} = -4.958$

4 { 3: $T_3(f,2)(x)$ <−1> each error; 1: approximation of $f(1.5)$ }

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \ne -5$.

2 { 1: value of Lagrange Error Bound; 1: explanation }

(c) $P(x) = T_4(g,0)(x)$

$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$

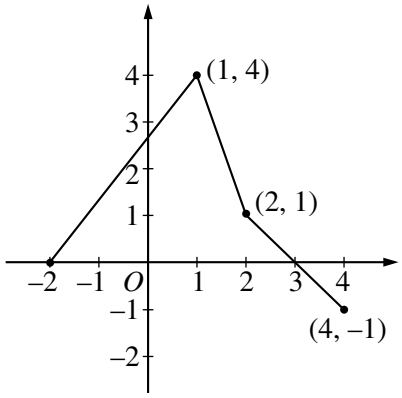
The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 { 2: $T_4(g,0)(x)$ <−1> each incorrect, missing, or extra term; 1: explanation }

Note:
<−1> max for improper use of $+\ldots$ or equality



5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

   (a) Compute $g(4)$ and $g(-2)$.

   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

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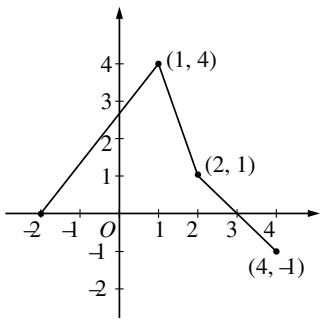
5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.



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   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$

2 { 1: $g(4)$; 1: $g(-2)$

(b) $g'(1) = f(1) = 4$

1: answer

(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$.

Therefore, g has absolute minimum at an endpoint of $[-2, 4]$.

Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$,

the absolute minimum value is -6.

3 { 1: interior analysis ; 1: endpoint analysis ; 1: answer

(d) One; $x = 1$

On $(-2, 1)$, $g''(x) = f'(x) > 0$

On $(1, 2)$, $g''(x) = f'(x) < 0$

On $(2, 4)$, $g''(x) = f'(x) < 0$

Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.

3 { 1: choice of $x = 1$ only ; 1: show $(1, g(1))$ is a point of inflection ; 1: show $(2, g(2))$ is not a point of inflection

6. Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \dfrac{1 + e^x}{x^2}$.

   (a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

   (b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

   (c) Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

END OF EXAMINATION

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6. Let f be the function whose graph goes through the point $(3,6)$ and whose derivative is given by $f'(x) = \dfrac{1+e^x}{x^2}$.
   (a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.
   (b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.
   (c) Use $\displaystyle\int_3^{3.1} f'(x)\,dx$ to evaluate $f(3.1)$.

(a) $f'(3) = \dfrac{1+e^3}{9} = 2.342$ or 2.343

$y - 6 = \dfrac{1+e^3}{9}(x-3)$

$y = 6 + \dfrac{1+e^3}{9}(x-3)$

$f(3.1) \approx 6 + \dfrac{1+e^3}{9}(0.1) = 6.234$

3
- 1: $f'(3)$
- 1: equation
- 1: approximation of $f(3.1)$

(b) $f(3.05) \approx f(3) + f'(3)(0.05)$

$= 6 + 0.11714 = 6.11714$

$f(3.1) \approx 6.11714 + f'(3.05)(0.05)$

$= 6.11714 + (2.37735)(0.05) = 6.236$

$f''(x) = \dfrac{x^2e^x - 2x(1+e^x)}{x^4} = \dfrac{(x-2)e^x - 2}{x^3}$

For $x \geq 3$, $f''(x) > \dfrac{e^x - 2}{x^3} > 0$ and the graph of f is concave upward on $(3, 3.1)$. Therefore, the Euler approximation lines at 3 and 3.05 lie below the graph.

4
- 1: Euler's method equations or equivalent table
- 1: Euler approximation to $f(3.1)$ (not eligible without first point)
- 1: $f''(x)$
- 1: reason

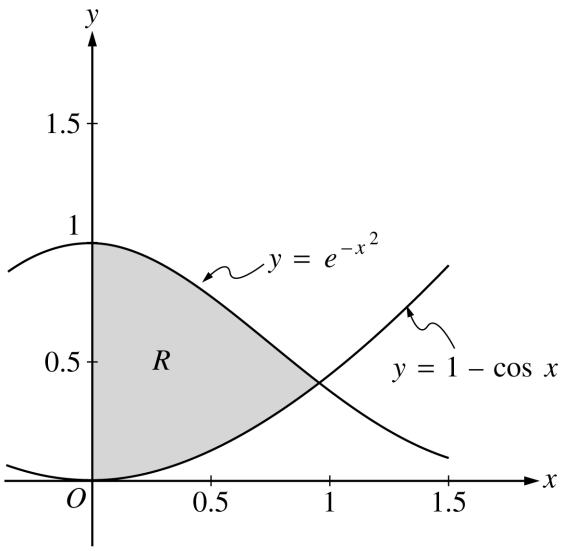
(c) $f(3.1) - f(3) = \displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx$

$f(3.1) = 6 + 0.2378 = 6.237$ or 6.238

2
- 1: $\displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx = f(3.1) - f(3)$
- 1: answer

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

   (a) Find the area of the region R.

   (b) Find the volume of the solid generated when the region R is revolved about the x-axis.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

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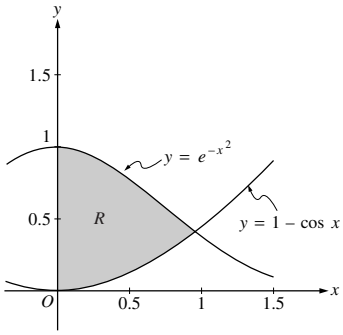

Region R

$e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$

(a) Area $= \int_0^A (e^{-x^2} - (1 - \cos x))\,dx$

$= 0.590$ or 0.591

2 { 1 : integrand; 1 : answer }

(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$

$= 0.55596\pi = 1.746$ or 1.747

3 { 2 : integrand and constant; $< -1 >$ each error; 1 : answer }

(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$

$= 0.461$

3 { 2 : integrand; $< -1 >$ each error; Note: 0/2 if not of the form $k\int_c^d (f(x) - g(x))^2\,dx$; 1 : answer }

1 : Correct limits in an integral in (a), (b), or (c).

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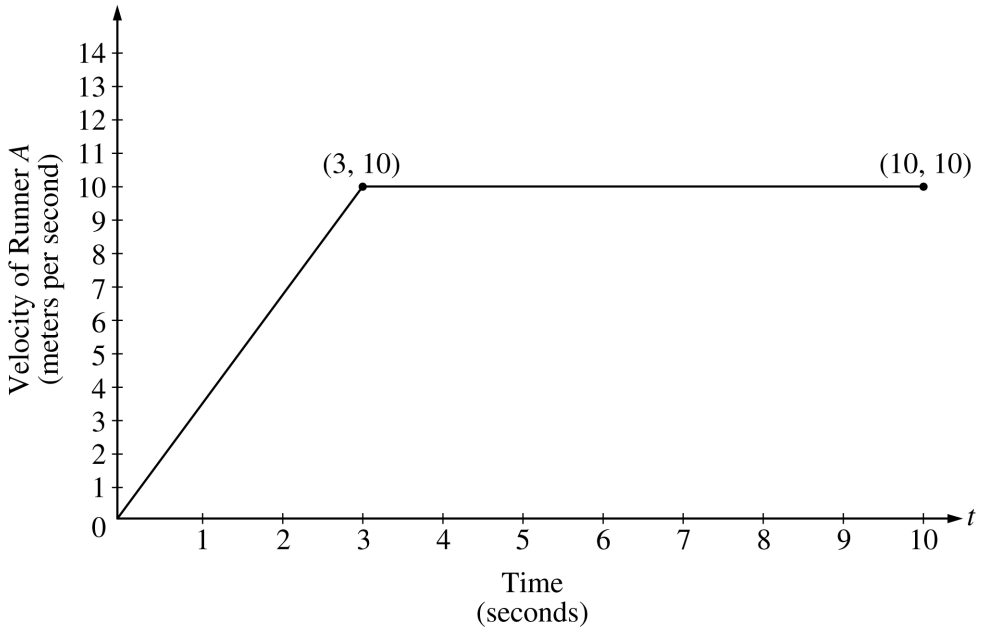

2. Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t + 3}$.

   (a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

   (b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

   (c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

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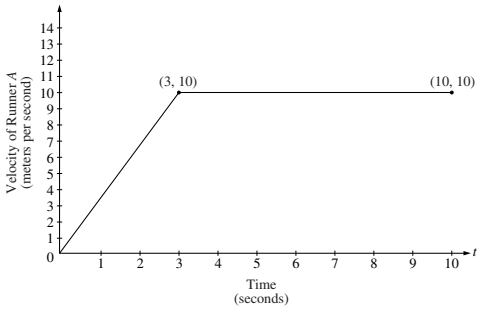

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(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Runner A: velocity $= \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$

$= 6.666$ or 6.667 meters/sec

Runner B: $v(2) = \dfrac{48}{7} = 6.857$ meters/sec

2 { 1 : velocity for Runner A ; 1 : velocity for Runner B }

(b) Runner A: acceleration $= \dfrac{10}{3} = 3.333$ meters/sec^2

Runner B: $a(2) = v'(2) = \left.\dfrac{72}{(2t+3)^2}\right|_{t=2}$

$= \dfrac{72}{49} = 1.469$ meters/sec^2

2 { 1 : acceleration for Runner A ; 1 : acceleration for Runner B }

(c) Runner A: distance $= \dfrac{1}{2}(3)(10) + 7(10) = 85$ meters

Runner B: distance $= \displaystyle\int_0^{10} \frac{24t}{2t+3}\,dt = 83.336$ meters

4 { 2 : distance for Runner A ; 1 : method ; 1 : answer ; 2 : distance for Runner B ; 1 : integral ; 1 : answer }

(units) meters/sec in part (a), meters/sec^2 in part (b), and meters in part (c), or equivalent.

1: units

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3. The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n + 2)}$, and $f(5) = \dfrac{1}{2}$.

   (a) Write the third-degree Taylor polynomial for f about $x = 5$.

   (b) Find the radius of convergence of the Taylor series for f about $x = 5$.

   (c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

END OF PART A OF SECTION II

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The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
<–1> each error or missing term

Note: <–1> max for improper use of extra terms, equality or +…

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n (x-5)^n}{2^n (n+2)}} \right| = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4
1 : general term
1 : sets up ratio test
1 : computes the limit
1 : applies ratio test to get radius of convergence

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$\left| f(6) - P_6(f,5)(6) \right| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2
1 : error bound $< \dfrac{1}{1000}$
1 : refers to an alternating series and indicates the error bound is found from the next term

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CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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4. A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is $(2,6)$, and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

   (a) Find the acceleration vector at time $t = 3$.

   (b) Find the position of the particle at time $t = 3$.

   (c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8 ?

   (d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

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(a) Find the acceleration vector at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

(a) acceleration vector $= \left(x''(t), y''(t)\right) = \left(\frac{2}{t^3}, -\frac{2}{t^3}\right)$

$\left(x''(3), y''(3)\right) = \left(\frac{2}{27}, -\frac{2}{27}\right)$

2 { 1 : components of acceleration vector as a function of t ; 1 : acceleration vector at $t = 3$ }

(b) $(x(t), y(t)) = \left(t + \frac{1}{t} + C_1,\ 2t - \frac{1}{t} + C_2\right)$

$(2,6) = (x(1), y(1)) = (2 + C_1, 1 + C_2)$

$C_1 = 0,\ C_2 = 5$

$(x(3), y(3)) = \left(3 + \frac{1}{3}, 6 - \frac{1}{3} + 5\right) = \left(\frac{10}{3}, \frac{32}{3}\right)$

3 { 1 : antidifferentiation ; 1 : uses initial condition at $t = 1$; 1 : position at $t = 3$ }

Note: max 1/3 [1–0–0] if no constants of integration

(c) $\frac{dy}{dx} = \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 8$

$2 + \frac{1}{t^2} = 8\left(1 - \frac{1}{t^2}\right); \quad t^2 = \frac{9}{6}$

$t = \sqrt{\frac{3}{2}}$

2 { 1 : $\frac{dy}{dx} = 8$ as equation in t ; 1 : solution for t }

(d) $\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 2$

– or –

Since $x(t) \to \infty$ as $t \to \infty$, the slope of the line is

$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{2t - \frac{1}{t} + 5}{t + \frac{1}{t}} = 2$

2 { 1 : considers limit of $\frac{dy}{dx}$ or $\frac{y(t)}{x(t)}$; 1 : answer }

Note: 0/2 if no consideration of limit

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

5. Consider the curve given by $xy^2 - x^3y = 6$.

   (a) Show that $\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$.

   (b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

   (c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

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Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

2 { 1 : implicit differentiation; 1 : verifies expression for $\dfrac{dy}{dx}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y - 3)(y + 2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9 - 9}{6 - 1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6 - 4}{-4 - 1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x - 1)$

4 { 1 : $y^2 - y = 6$; 1 : solves for y; 2 : tangent lines

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

3 { 1 : sets denominator of $\dfrac{dy}{dx}$ equal to 0; 1 : substitutes $y = \dfrac{1}{2}x^2$ or $x = \pm\sqrt{2y}$ into the equation for the curve; 1 : solves for x-coordinate

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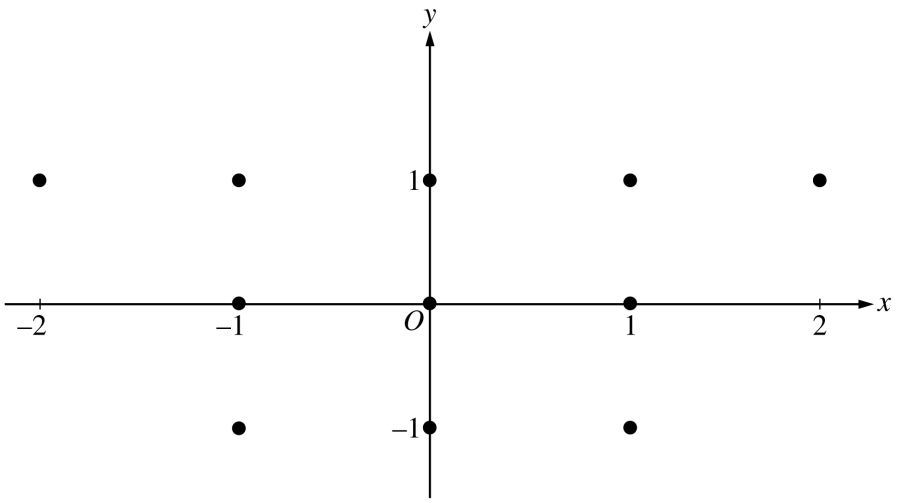
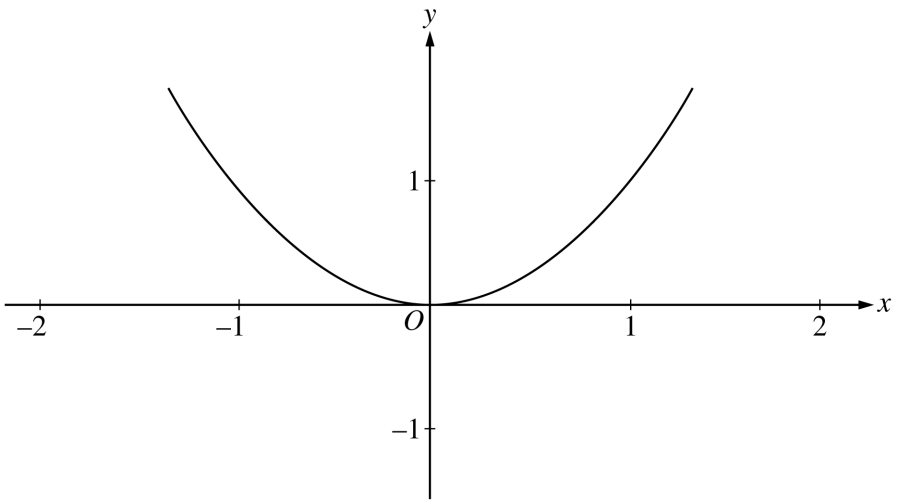
6. Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.

(Note: Use the axes provided in the pink test booklet.)



(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.



(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).

END OF EXAMINATION

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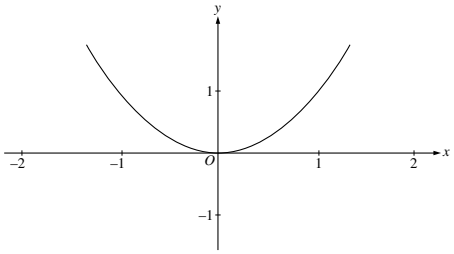
Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.

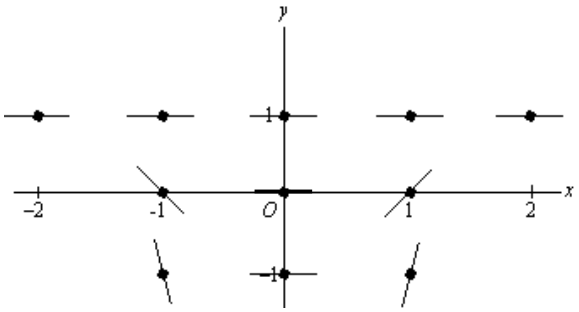
(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).



(a)



2 { 1 : zero slope at 7 points with $y = 1$ and $x = 0$; 1 : negative slope at $(-1,0)$ and $(-1,-1)$, positive slope at $(1,0)$ and $(1,-1)$, steeper slope at $y = -1$ than $y = 0$ }

(b) The graph does not have slope 0 where $y = 1$.
– or –
The slope field shown suggests that solutions are asymptotic to $y = 1$ from below, but the graph does not exhibit this behavior.

1 : reason

(c) $\frac{1}{(y-1)^2}\,dy = x\,dx$

$-\frac{1}{y-1} = \frac{1}{2}x^2 + C$

$\frac{1}{2} = 0 + C\,; \quad C = \frac{1}{2}$

$-\frac{1}{y-1} = \frac{1}{2}\left(x^2+1\right)\,; \quad y = 1 - \frac{2}{x^2+1}$

5 { 1 : separates variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition $f(0) = -1$; 1 : solves for y; 0/1 if y is linear }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(d) range is $-1 \le y < 1$

1 : answer
0/1 if –1 not in range

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CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \cos\left(t^3\right) \text{ and } \frac{dy}{dt} = 3\sin\left(t^2\right)$$

for $0 \le t \le 3$. At time $t = 2$, the object is at position $(4, 5)$.

(a) Write an equation for the line tangent to the curve at $(4, 5)$.

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 3$.

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Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \cos\left(t^3\right) \text{ and } \frac{dy}{dt} = 3\sin\left(t^2\right)$$

for $0 \le t \le 3$. At time $t = 2$, the object is at position (4,5).

(a) Write an equation for the line tangent to the curve at (4,5).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 3$.

(a) $\dfrac{dy}{dx} = \dfrac{3\sin\left(t^2\right)}{\cos\left(t^3\right)}$

$\left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{3\sin\left(2^2\right)}{\cos\left(2^3\right)} = 15.604$

$y - 5 = 15.604(x - 4)$

1 : tangent line

(b) Speed $= \sqrt{\cos^2(8) + 9\sin^2(4)} = 2.275$

1 : answer

(c) Distance $= \displaystyle\int_0^1 \sqrt{\cos^2\left(t^3\right) + 9\sin^2\left(t^2\right)}\, dt$

$= 1.458$

3 : 2 : distance integral
$< -1 >$ each integrand error
$< -1 >$ error in limits
1 : answer

(d) $x(3) = 4 + \displaystyle\int_2^3 \cos\left(t^3\right) dt = 3.953$ or 3.954

$y(3) = 5 + \displaystyle\int_2^3 3\sin\left(t^2\right) dt = 4.906$

4 : 1 : definite integral for x
1 : answer for $x(3)$
1 : definite integral for y
1 : answer for $y(3)$

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t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

2. The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

   (a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

   (b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

   (c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

   (d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

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Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$W'(12) \approx \dfrac{W(15) - W(12)}{15 - 12} = -\dfrac{1}{3}$ °C/day or

$W'(12) \approx \dfrac{W(12) - W(9)}{12 - 9} = -\dfrac{2}{3}$ °C/day or

$W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = -\dfrac{1}{2}$ °C/day

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer (with units)} \end{cases}$

(b) $\dfrac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \dfrac{1}{15}(376.5) = 25.1\,°\text{C}$

$2 : \begin{cases} 1 : \text{trapezoidal method} \\ 1 : \text{answer} \end{cases}$

(c) $P'(12) = 10e^{-t/3} - \dfrac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

$2 : \begin{cases} 1 : P'(12) \text{ (with or without units)} \\ 1 : \text{interpretation} \end{cases}$

(d) $\dfrac{1}{15}\displaystyle\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757\,°\text{C}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and} \\ \quad \text{average value constant} \\ 1 : \text{answer} \end{cases}$

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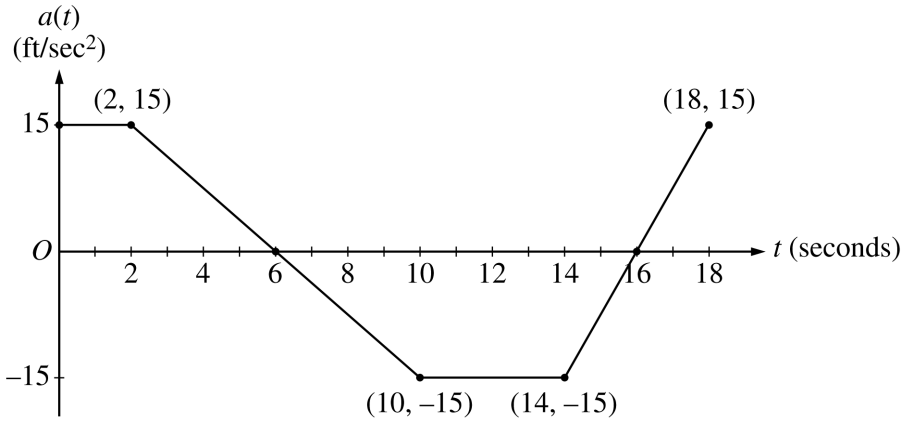

3. A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.

   (a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

   (b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

   (c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

   (d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

END OF PART A OF SECTION II

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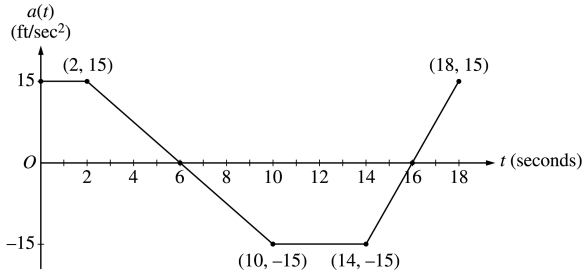
Question 3

A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.



(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

1 : answer and reason

(b) At time $t = 12$ because
$v(12) - v(0) = \int_0^{12} a(t)\,dt = 0$.

2 : { 1 : $t = 12$; 1 : reason }

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.
The absolute maximum must occur at $t = 6$ or at an endpoint.
$v(6) = 55 + \int_0^6 a(t)\,dt$
$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$
$\int_6^{18} a(t)\,dt < 0$ so $v(18) < v(6)$

4 : { 1 : $t = 6$; 1 : absolute maximum velocity; 1 : identifies $t = 6$ and $t = 18$ as candidates or indicates that v increases, decreases, then increases; 1 : eliminates $t = 18$ }

(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where
$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0$.

2 : { 1 : answer; 1 : reason }

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

---

4. Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

   (a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

   (b) On what intervals, if any, is the graph of h concave up? Justify your answer.

   (c) Write an equation for the line tangent to the graph of h at $x = 4$.

   (d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

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Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

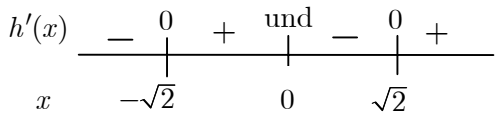
(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$



Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

4 : { 1 : $x = \pm\sqrt{2}$; 1 : analysis; 2 : conclusions; $< -1 >$ not dealing with discontinuity at 0

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

3 : { 1 : $h''(x)$; 1 : $h''(x) > 0$; 1 : answer

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

---

5. Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x \to \infty} f(x) = 0$.

   (a) Evaluate $\int_1^{\infty} -3xf(x)\,dx$. Show the work that leads to your answer.

   (b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

   (c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

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Question 5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Evaluate $\int_1^\infty -3xf(x)\,dx$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

(c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

(a) $\int_1^\infty -3xf(x)\,dx$

$= \int_1^\infty f'(x)\,dx = \lim_{b\to\infty}\int_1^b f'(x)\,dx = \lim_{b\to\infty} f(x)\Big|_1^b$

$= \lim_{b\to\infty} f(b) - f(1) = 0 - 4 = -4$

2 : { 1 : use of FTC; 1 : answer from limiting process }

(b) $f(1.5) \approx f(1) + f'(1)(0.5)$

$= 4 - 3(1)(4)(0.5) = -2$

$f(2) \approx -2 + f'(1.5)(0.5)$

$\approx -2 - 3(1.5)(-2)(0.5) = 2.5$

2 : { 1 : Euler's method equations or equivalent table; 1 : Euler approximation to $f(2)$ (not eligible without first point) }

(c) $\frac{1}{y}\,dy = -3x\,dx$

$\ln y = -\frac{3}{2}x^2 + k$

$y = Ce^{-3/2\,x^2}$

$4 = Ce^{-3/2}$; $C = 4e^{3/2}$

$y = 4e^{3/2}\,e^{-3/2\,x^2}$

5 : { 1 : separates variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition $f(1) = 4$; 1 : solves for y }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

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6. A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim_{x \to 0} \dfrac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

END OF EXAMINATION

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Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim_{x\to 0} \dfrac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

(a) $\lim_{n\to\infty} \left| \dfrac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \dfrac{(n+2)}{(n+1)}\dfrac{x}{3} \right| = \left| \dfrac{x}{3} \right| < 1$

At $x = -3$, the series is $\sum_{n=0}^{\infty} (-1)^n \dfrac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\sum_{n=0}^{\infty} \dfrac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 : { 1 : sets up ratio test; 1 : computes limit; 1 : conclusion of ratio test; 1 : endpoint conclusion }

(b) $\lim_{x\to 0} \dfrac{f(x) - \frac{1}{3}}{x} = \lim_{x\to 0} \left(\dfrac{2}{3^2} + \dfrac{3}{3^3}x + \dfrac{4}{3^4}x^2 + \cdots \right) = \dfrac{2}{9}$

1 : answer

(c) $\int_0^1 f(x)\,dx = \int_0^1 \left(\dfrac{1}{3} + \dfrac{2}{3^2}x + \dfrac{3}{3^3}x^2 + \cdots + \dfrac{n+1}{3^{n+1}}x^n + \cdots \right) dx$

$= \left(\dfrac{1}{3}x + \dfrac{1}{3^2}x^2 + \dfrac{1}{3^3}x^3 + \cdots + \dfrac{1}{3^{n+1}}x^{n+1} + \cdots \right) \Big|_{x=0}^{x=1}$

$= \dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \cdots + \dfrac{1}{3^{n+1}} + \cdots$

3 : { 1 : antidifferentiation of series; 1 : first three terms for definite integral series; 1 : general term }

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\int_0^1 f(x)\,dx = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}} = \dfrac{1}{2}$.

1 : answer

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

   (a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

   (b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

   (c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

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AP® CALCULUS AB 2002 SCORING GUIDELINES

Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2 \right) dx$

$= 7.515\pi$ or 23.609

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$

$h(0.5) = 2.3418$

$h(1) = 2.718$

The absolute minimum is 2.330.

The absolute maximum is 2.718.

2 { 1 : integral
1 : answer

4 { 1 : limits and constant
2 : integrand
< −1 > each error
Note: 0 / 2 if not of the form $k \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$
1 : answer

3 { 1 : considers $h'(x) = 0$
1 : identifies critical point and endpoints as candidates
1 : answers

Note: Errors in computation come off the third point.

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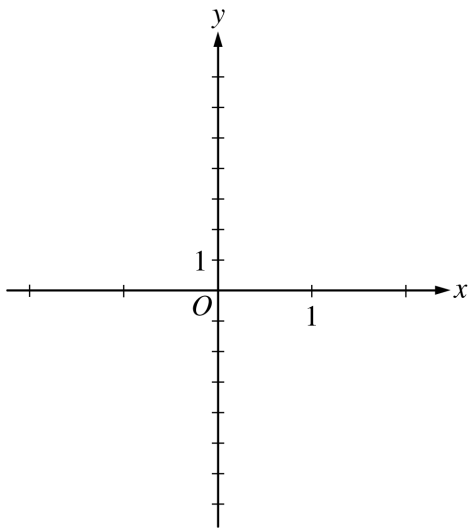
CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

   (a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path. **(Note: Use the axes provided in the test booklet.)**



   (b) Find the range of $x(t)$ and the range of $y(t)$.

   (c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

   (d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π ? Justify your answer.

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AP® CALCULUS BC
2002 SCORING GUIDELINES (Form B)

Question 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

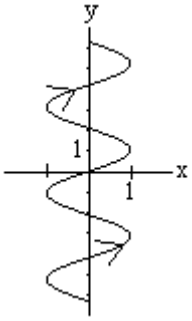
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

(a)



2 { 1 : graph; three cycles of sine; x between -1 and 1; y between -2π and 2π / 1 : direction }

(b) $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

2 { 1 : closed interval for $x(t)$ / 1 : closed interval for $y(t)$ }

(c) $x'(t) = 3\cos 3t = 0$

$3t = \frac{\pi}{2}$; $t = \frac{\pi}{6}$

Speed $= \sqrt{9\cos^2(3t) + 4}$

At $t = \frac{\pi}{6}$,

Speed $= \sqrt{9\cos^2\left(\frac{\pi}{2}\right) + 4} = 2$

3 { 1 : $x'(t) = 3\cos 3t = 0$ / 1 : solves for t / 1 : speed at student's time }

(d) Distance $= \int_{-\pi}^{\pi} \sqrt{9\cos^2(3t) + 4}\, dt$

$= 17.973 > 5\pi$

2 { 1 : integral for distance / 1 : conclusion with justification }

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2. The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round your answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$, and explain the meaning of $H(17)$ and $H'(17)$ in the context of the amusement park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

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Question 2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

(a) $\int_9^{17} E(t)\,dt = 6004.270$

6004 people entered the park by 5 pm.

3 { 1 : limits; 1 : integrand; 1 : answer }

(b) $15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$

The amount collected was \$104,048.

or

$\int_{17}^{23} E(t)\,dt = 1271.283$

1271 people entered the park between 5 pm and 11 pm, so the amount collected was $\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041$.

1 : setup

(c) $H'(17) = E(17) - L(17) = -380.281$

There were 3725 people in the park at $t = 17$.

The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.

3 { 1 : value of $H'(17)$; 2 : meanings { 1 : meaning of $H(17)$; 1 : meaning of $H'(17)$; $< -1 >$ if no reference to $t = 17$ } }

(d) $H'(t) = E(t) - L(t) = 0$

$t = 15.794$ or 15.795

2 { 1 : $E(t) - L(t) = 0$; 1 : answer }

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

2. The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

   (a) Is the amount of pollutant increasing at time $t = 9$? Why or why not?

   (b) For what value of t will the number of gallons of pollutant be at its minimum? Justify your answer.

   (c) Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

   (d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

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GO ON TO THE NEXT PAGE.

AP® CALCULUS AB
2002 SCORING GUIDELINES (Form B)

Question 2

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(a) Is the amount of pollutant increasing at time $t = 9$? Why or why not?

(b) For what value of t will the number of gallons of pollutant be at its minimum? Justify your answer.

(c) Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

(a) $P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$
so the amount is not increasing at this time.

1 : answer with reason

(b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0$
$t = (5\ln 3)^2 = 30.174$
$P'(t)$ is negative for $0 < t < (5\ln 3)^2$ and positive for $t > (5\ln 3)^2$. Therefore there is a minimum at $t = (5\ln 3)^2$.

3 { 1 : sets $P'(t) = 0$; 1 : solves for t ; 1 : justification }

(c) $P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt$
$= 35.104 < 40$, so the lake is safe.

3 { 1 : integrand ; 1 : limits ; 1 : conclusion with reason based on integral of $P'(t)$ }

(d) $P'(0) = 1 - 3 = -2$. The lake will become safe when the amount decreases by 10. A linear model predicts this will happen when $t = 5$.

2 { 1 : slope of tangent line ; 1 : answer }

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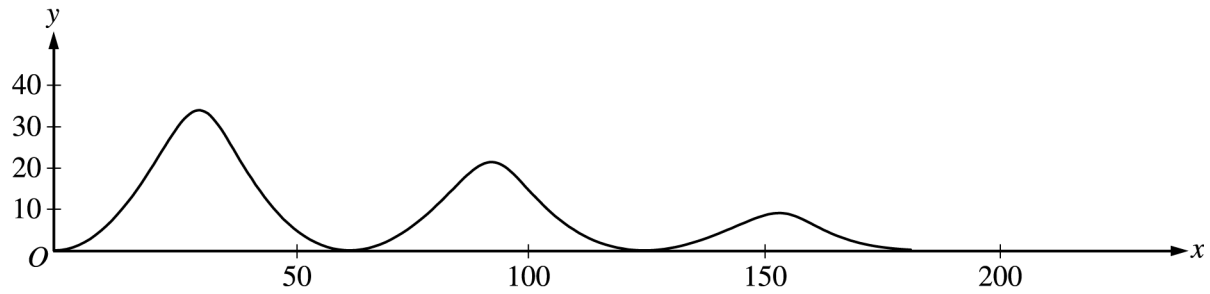

3. The figure above shows the path traveled by a roller coaster car over the time interval $0 \le t \le 18$ seconds. The position of the car at time t seconds can be modeled parametrically by

$$x(t) = 10t + 4 \sin t$$
$$y(t) = (20 - t)(1 - \cos t),$$

where x and y are measured in meters. The derivatives of these functions are given by

$$x'(t) = 10 + 4 \cos t$$
$$y'(t) = (20 - t) \sin t + \cos t - 1.$$

(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.

(c) Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.

(d) For $0 < t < 18$, there are two times at which the car is at ground level ($y = 0$). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

END OF PART A OF SECTION II

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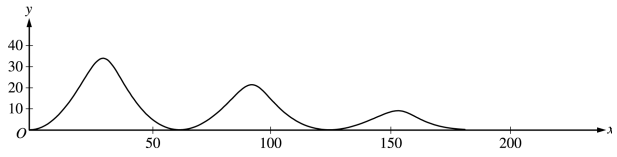
Question 3

The figure above shows the path traveled by a roller coaster car over the time interval $0 \le t \le 18$ seconds. The position of the car at time t seconds can be modeled parametrically by $x(t) = 10t + 4\sin t,\ y(t) = (20 - t)(1 - \cos t)$, where x and y are measured in meters. The derivatives of these functions are given by $x'(t) = 10 + 4\cos t,\ y'(t) = (20 - t)\sin t + \cos t - 1$.



(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.

(c) Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.

(d) For $0 < t < 18$, there are two times at which the car is at ground level ($y = 0$). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

(a) $\text{Slope} = \left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{y'(2)}{x'(2)} = \dfrac{18\sin 2 + \cos 2 - 1}{10 + 4\cos 2}$
$= 1.793$ or 1.794

1 : answer using $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$

(b) $x(t) = 10t + 4\sin t = 140;\ t_0 = 13.647083$
$x''(t_0) = -3.529,\ y''(t_0) = 1.225$ or 1.226
Acceleration vector is $<-3.529, 1.225>$
or $<-3.529, 1.226>$

2 : 1 : identifies acceleration vector as derivative of velocity vector
1 : computes acceleration vector when $x = 140$

(c) $y'(t) = (20 - t)\sin t + \cos t - 1 = 0$
$t_1 = 3.023$ or 3.024 at maximum height
$\text{Speed} = \sqrt{\left(x'(t_1)\right)^2 + \left(y'(t_1)\right)^2} = \left|x'(t_1)\right|$
$= 6.027$ or 6.028

3 : 1 : sets $y'(t) = 0$
1 : selects first $t > 0$
1 : speed

(d) $y(t) = 0$ when $t = 2\pi$ and $t = 4\pi$
$\text{Average speed} = \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$
$= \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos t)^2 + \left((20 - t)\sin t + \cos t - 1\right)^2}\,dt$

3 : 1 : $t = 2\pi, t = 4\pi$
1 : limits and constant
1 : integrand

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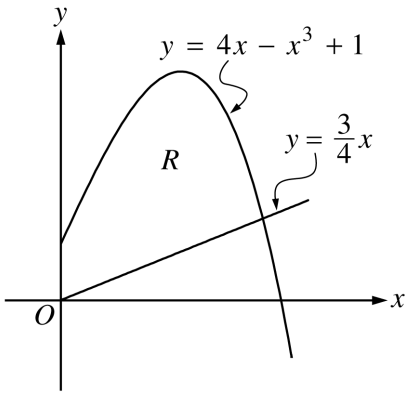

3. Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = 4x - x^3 + 1$ and $y = \frac{3}{4}x$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the x-axis.

   (c) Write an expression involving one or more integrals that gives the perimeter of R. Do not evaluate.

END OF PART A OF SECTION II

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AP® CALCULUS BC
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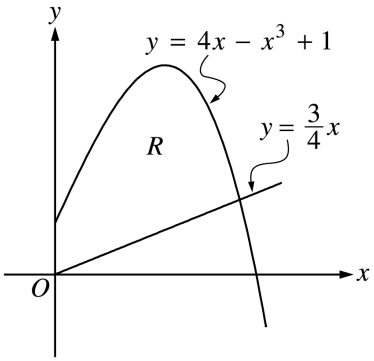
Question 3

Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = 4x - x^3 + 1$ and $y = \frac{3}{4}x$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Write an expression involving one or more integrals that gives the perimeter of R. Do not evaluate.



Region R

$4x - x^3 + 1 = \frac{3}{4}x$ when $x = 1.94045 = A$

(a) Area $= \int_0^A \left(4x - x^3 + 1 - \frac{3}{4}x\right)dx$

$= 4.514$ or 4.515

3 { 1 : limits; 1 : integrand; 1 : answer }

(b) Volume

$= \pi\int_0^A \left(\left(4x - x^3 + 1\right)^2 - \left(\frac{3}{4}x\right)^2\right)dx$

$= 18.291\pi$ or 57.463

3 { 1 : limits and constant; 1 : integrand; 1 : answer }

(c) Perimeter $= 1 + \sqrt{(1.940)^2 + (1.455)^2} + \int_0^A \sqrt{1 + (4 - 3x^2)^2}\,dx$

3 { 1 : uses $y' = 4 - 3x^2$ in integrand; 1 : arc length integral; 1 : answer }

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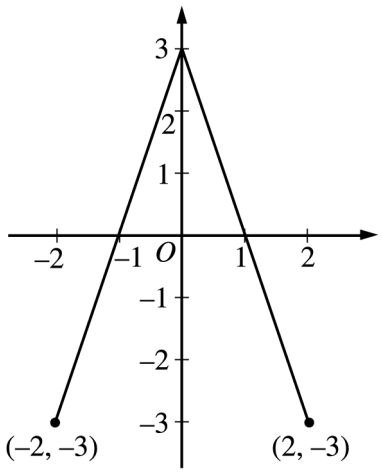
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f

4. The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

   (a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

   (b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

   (c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

   (d) On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.
   (Note: The axes are provided in the pink test booklet only.)

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Question 4

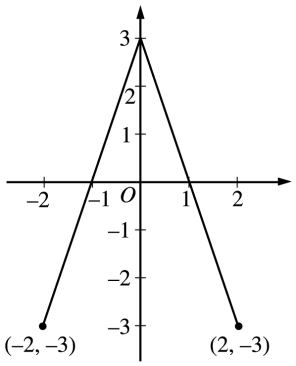
The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2,2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2,2)$ is the graph of g concave down? Explain your reasoning.

(d) On the axes provided, sketch the graph of g on the closed interval $[-2,2]$.



Graph of f

(a) $g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\frac{3}{2}$

$g'(-1) = f(-1) = 0$

$g''(-1) = f'(-1) = 3$

3 { 1 : $g(-1)$; 1 : $g'(-1)$; 1 : $g''(-1)$

(b) g is increasing on $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval.

2 { 1 : interval; 1 : reason

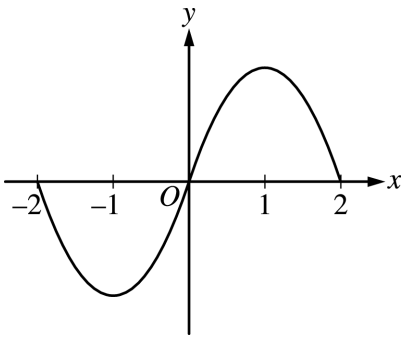
(c) The graph of g is concave down on $0 < x < 2$ because $g''(x) = f'(x) < 0$ on this interval.

or

because $g'(x) = f(x)$ is decreasing on this interval.

2 { 1 : interval; 1 : reason

(d)



2 { 1 : $g(-2) = g(0) = g(2) = 0$; 1 : appropriate increasing/decreasing and concavity behavior; $< -1 >$ vertical asymptote

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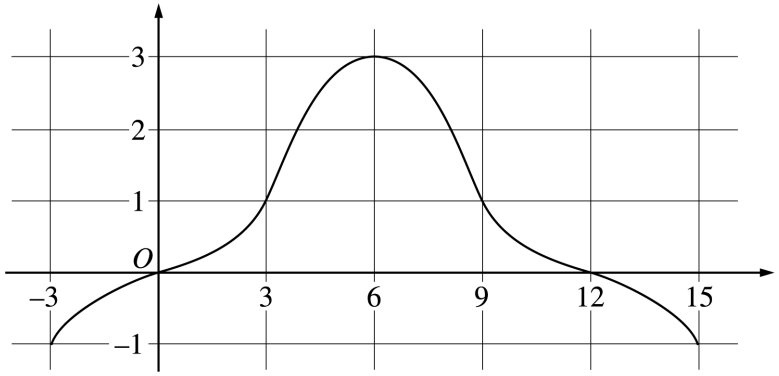
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f

4. The graph of a differentiable function f on the closed interval $[-3, 15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.

   (a) Find $g(6)$, $g'(6)$, and $g''(6)$.

   (b) On what intervals is g decreasing? Justify your answer.

   (c) On what intervals is the graph of g concave down? Justify your answer.

   (d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

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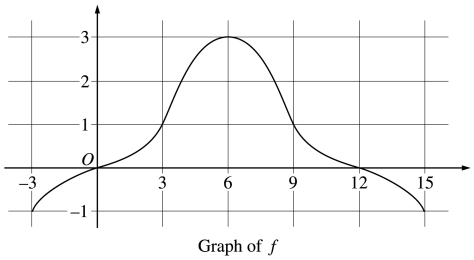
Question 4

The graph of a differentiable function f on the closed interval $[-3,15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.



Graph of f

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

3 { 1 : $g(6)$; 1 : $g'(6)$; 1 : $g''(6)$ }

(b) g is decreasing on $[-3,0]$ and $[12,15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

3 { 1 : $[-3,0]$; 1 : $[12,15]$; 1 : justification }

(c) The graph of g is concave down on $(6,15)$ since $g' = f$ is decreasing on this interval.

2 { 1 : interval; 1 : justification }

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$

$= 12$

1 : trapezoidal method

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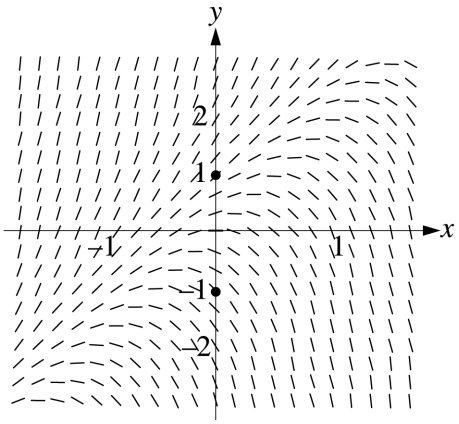
5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.
(Note: Use the slope field provided in the pink test booklet.)



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

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Question 5

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

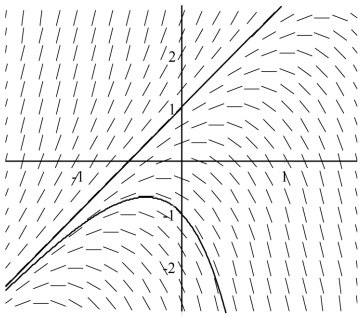
(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.

(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(a)



2 { 1 : solution curve through $(0,1)$
1 : solution curve through $(0,-1)$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$
$= 1 + (2 - 0)(0.1) = 1.2$
$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$
$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

2 { 1 : Euler's method equations or equivalent table applied to (at least) two iterations
1 : Euler approximation to $f(0.2)$ (not eligible without first point)

(c) Substitute $y = 2x + b$ in the DE:
$2 = 2(2x + b) - 4x = 2b$, so $b = 1$
OR
Guess $b = 1$, $y = 2x + 1$
Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

2 { 1 : uses $\frac{d}{dx}(2x + b) = 2$ in DE
1 : $b = 1$

(d) g has local maximum at $(0,0)$.
$g'(0) = \left.\frac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and
$g''(x) = \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$, so
$g''(0) = 2\,g'(0) - 4 = -4 < 0$.

3 { 1 : $g'(0) = 0$
1 : shows $g''(0) = -4$
1 : conclusion

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

5. Consider the differential equation $\frac{dy}{dx} = \frac{3 - x}{y}$.

   (a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

   (b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

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AP® CALCULUS AB
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Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\frac{dy}{dx} = 0$ when $x = 3$

$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2}$,

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\frac{dy}{dx}$ is negative to the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

3 { 1 : $x = 3$; 1 : local minimum ; 1 : justification }

(b) $y\,dy = (3-x)\,dx$

$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$

$8 = 18 - 18 + C$; $C = 8$

$y^2 = 6x - x^2 + 16$

$y = -\sqrt{6x - x^2 + 16}$

6 { 1 : separates variables ; 1 : antiderivative of dy term ; 1 : antiderivative of dx term ; 1 : constant of integration ; 1 : uses initial condition $g(6) = -4$; 1 : solves for y }

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

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6. The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

END OF EXAMINATION

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Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a) $\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} 2x \right| = \left| 2x \right|$

$\left| 2x \right| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \le x < \frac{1}{2}$.

5 { 1 : sets up ratio; 1 : computes limit of ratio; 1 : identifies interior of interval of convergence; 2 : analysis/conclusion at endpoints (1 : right endpoint, 1 : left endpoint); $< -1 >$ if endpoints not $x = \pm\frac{1}{2}$; $< -1 >$ if multiple intervals }

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$

2 { 1 : first 4 terms; 1 : general term }

(c) The series in (b) is a geometric series.

$$f'\left(-\frac{1}{3}\right) = 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \ldots + 2\left(2 \cdot \left(-\frac{1}{3}\right)\right)^n + \ldots$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + 2\left(-\frac{2}{3}\right)^n + \ldots$$

$$= \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

OR

$f'(x) = \frac{2}{1-2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$f'\left(-\frac{1}{3}\right) = \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$

2 { 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form; 1 : answer for student's series }

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6. The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

   (a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

   (b) Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

   (c) Give a value of p such that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

   (d) Give a value of p such that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

END OF EXAMINATION

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2002 SCORING GUIDELINES (Form B)

Question 6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$= \sum_{n=1}^{\infty}\frac{(-3x)^n}{n}$ or $\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n}x^n$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

2 { 1 : series; 1 : interval of convergence }

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges for $2p \le 1$.

3 { 1 : correct p; 1 : reason why $\sum\frac{(-1)^n}{n^p}$ converges; 1 : reason why $\sum\frac{1}{n^{2p}}$ diverges }

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges for $2p > 1$.

3 { 1 : correct p; 1 : reason why $\sum\frac{1}{n^p}$ diverges; 1 : reason why $\sum\frac{1}{n^{2p}}$ converges }

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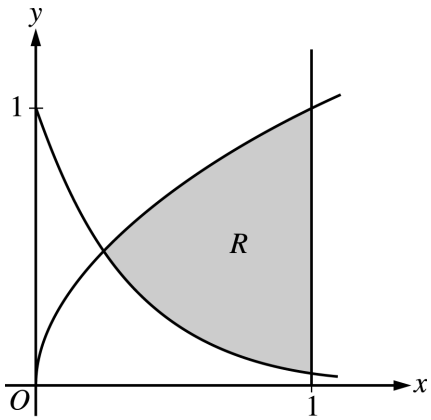
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

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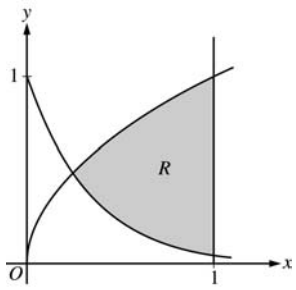
Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.



Point of intersection
$e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$

(a) Area $= \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx$
$= 0.442$ or 0.443

1: Correct limits in an integral in (a), (b), or (c)

2 : { 1 : integrand; 1 : answer }

(b) Volume $= \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right) dx$
$= 0.453\pi$ or 1.423 or 1.424

3 : { 2 : integrand
$< -1 >$ reversal
$< -1 >$ error with constant
$< -1 >$ omits 1 in one radius
$< -2 >$ other errors
1 : answer }

(c) Length $= \sqrt{x} - e^{-3x}$
Height $= 5\left(\sqrt{x} - e^{-3x}\right)$

Volume $= \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$

3 : { 2 : integrand
$< -1 >$ incorrect but has $\sqrt{x} - e^{-3x}$ as a factor
1 : answer }

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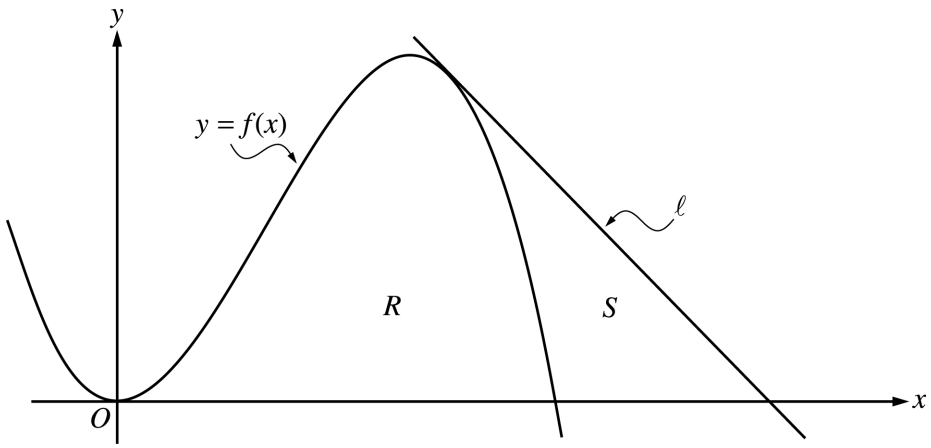
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

   (a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

   (b) Find the area of S.

   (c) Find the volume of the solid generated when R is revolved about the x-axis.

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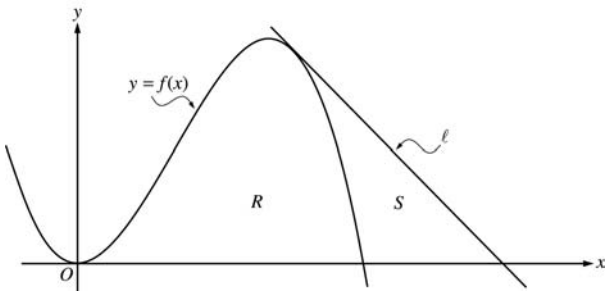
2003 SCORING GUIDELINES (Form B)

Question 1

Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.



(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$
$f(3) = 36 - 27 = 9$
Tangent line at $x = 3$ is
$y = -3(x - 3) + 9 = -3x + 18$,
which is the equation of line ℓ.

2 : { 1 : finds $f'(3)$ and $f(3)$; 1 : { finds equation of tangent line or shows (3,9) is on both the graph of f and line ℓ }

(b) $f(x) = 0$ at $x = 4$
The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx$$
$$= 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12)$$
$$= 7.916 \text{ or } 7.917$$

4 : { 2 : integral for non-triangular region (1 : limits ; 1 : integrand) ; 1 : area of triangular region ; 1 : answer }

(c) $\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$
$= 156.038\pi$ or 490.208

3 : { 1 : limits and constant ; 1 : integrand ; 1 : answer }

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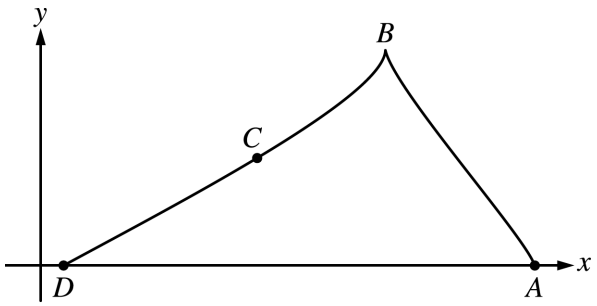

2. A particle starts at point A on the positive x-axis at time $t = 0$ and travels along the curve from A to B to C to D, as shown above. The coordinates of the particle's position $(x(t), y(t))$ are differentiable functions of t, where $x'(t) = \frac{dx}{dt} = -9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \frac{dy}{dt}$ is not explicitly given. At time $t = 9$, the particle reaches its final position at point D on the positive x-axis.

   (a) At point C, is $\frac{dy}{dt}$ positive? At point C, is $\frac{dx}{dt}$ positive? Give a reason for each answer.

   (b) The slope of the curve is undefined at point B. At what time t is the particle at point B?

   (c) The line tangent to the curve at the point $(x(8), y(8))$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

   (d) How far apart are points A and D, the initial and final positions, respectively, of the particle?

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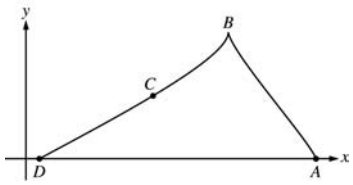
2003 SCORING GUIDELINES

Question 2

A particle starts at point A on the positive x-axis at time $t = 0$ and travels along the curve from A to B to C to D, as shown above. The coordinates of the particle's position $\big(x(t),\, y(t)\big)$ are differentiable functions of t, where $x'(t) = \dfrac{dx}{dt} = -9\cos\left(\dfrac{\pi t}{6}\right)\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \dfrac{dy}{dt}$ is not explicitly given.



At time $t = 9$, the particle reaches its final position at point D on the positive x-axis.

(a) At point C, is $\dfrac{dy}{dt}$ positive? At point C, is $\dfrac{dx}{dt}$ positive? Give a reason for each answer.

(b) The slope of the curve is undefined at point B. At what time t is the particle at point B?

(c) The line tangent to the curve at the point $\big(x(8),\, y(8)\big)$ has equation $y = \dfrac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

(d) How far apart are points A and D, the initial and final positions, respectively, of the particle?

(a) At point C, $\dfrac{dy}{dt}$ is not positive because $y(t)$ is decreasing along the arc BD as t increases.
At point C, $\dfrac{dx}{dt}$ is not positive because $x(t)$ is decreasing along the arc BD as t increases.

$2 :$ { $1 :$ $\dfrac{dy}{dt}$ not positive with reason; $1 :$ $\dfrac{dx}{dt}$ not positive with reason }

(b) $\dfrac{dx}{dt} = 0$; $\cos\left(\dfrac{\pi t}{6}\right) = 0$ or $\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right) = 0$
$\dfrac{\pi t}{6} = \dfrac{\pi}{2}$ or $\dfrac{\pi\sqrt{t+1}}{2} = \pi$; $t = 3$ for both.
Particle is at point B at $t = 3$.

$2 :$ { $1 :$ sets $\dfrac{dx}{dt} = 0$; $1 : t = 3$ }

(c) $x'(8) = -9\cos\left(\dfrac{4\pi}{3}\right)\sin\left(\dfrac{3\pi}{2}\right) = -\dfrac{9}{2}$
$\dfrac{y'(8)}{x'(8)} = \dfrac{dy}{dx} = \dfrac{5}{9}$
$y'(8) = \dfrac{5}{9}x'(8) = -\dfrac{5}{2}$
The velocity vector is $< -4.5, -2.5 >$.
Speed $= \sqrt{4.5^2 + 2.5^2} = 5.147$ or 5.148

$3 :$ { $1 : x'(8)$; $1 : y'(8)$; $1 :$ speed }

(d) $x(9) - x(0) = \displaystyle\int_0^9 x'(t)\,dt$
$= -39.255$
The initial and final positions are 39.255 apart.

$2 :$ { $1 :$ integral; $1 :$ answer }

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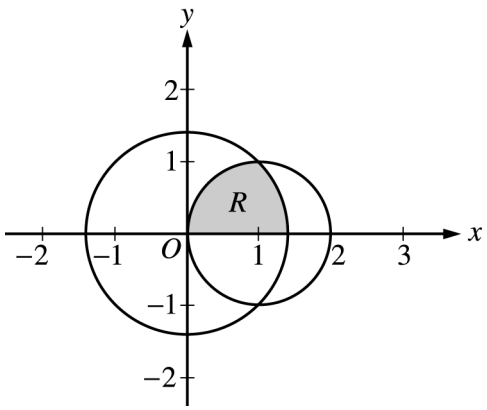

2. The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1, 1)$ and $(1, -1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.

   (a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

   (b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

   (c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

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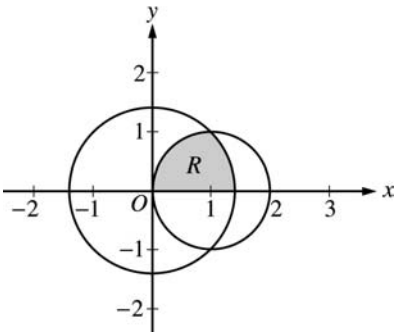
AP® CALCULUS BC
2003 SCORING GUIDELINES (Form B)

Question 2



The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.

(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) $\text{Area} = \displaystyle\int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

OR

$\text{Area} = \dfrac{1}{4}\left(\pi \cdot 1^2\right) + \displaystyle\int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

3 : { 1 : integrand for larger circle; 1 : integrand or geometric area for smaller circle; 1 : limits on integral(s) }

Note: < –1 > if no addition of terms

(b) $\text{Area} = \displaystyle\int_0^1 \left(\sqrt{2-y^2} - \left(1 - \sqrt{1-y^2}\right)\right) dy$

3 : { 1 : limits; 2 : integrand; < −1 > reversal; < −1 > algebra error in solving for x; < −1 > add rather than subtract; < −2 > other errors }

(c) $\text{Area} = \displaystyle\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR

$\text{Area} = \dfrac{1}{8}\pi\left(\sqrt{2}\right)^2 + \displaystyle\int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

3 : { 1 : integrand or geometric area for larger circle; 1 : integrand for smaller circle; 1 : limits on integral(s) }

Note: < –1 > if no addition of terms

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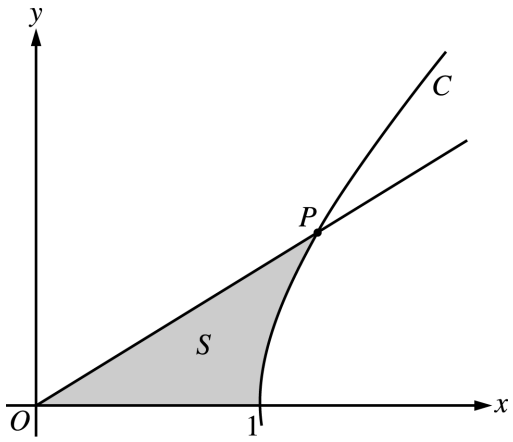

3. The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

   (a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

   (b) Set up and evaluate an integral expression with respect to y that gives the area of S.

   (c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

   (d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

END OF PART A OF SECTION II

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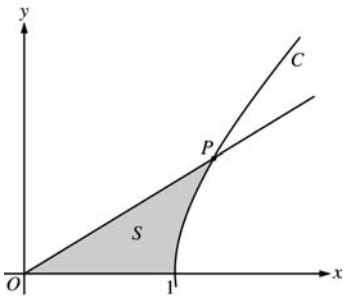
2003 SCORING GUIDELINES

Question 3



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.
Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}$. At P, $\frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}$.

$2 :$ 1 : coordinates of P; 1 : $\frac{dx}{dy}$ at P

(b) Area $= \int_0^{3/4}\left(\sqrt{1+y^2} - \frac{5}{3}y\right)dy$
$= 0.346$ or 0.347

$3 :$ 1 : limits; 1 : integrand; 1 : answer

(c) $x = r\cos\theta$; $y = r\sin\theta$
$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$
$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$

$2 :$ 1 : substitutes $x = r\cos\theta$ and $y = r\sin\theta$ into $x^2 - y^2 = 1$; 1 : isolates r^2

(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

Area $= \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\,d\theta$
$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\,d\theta$

$2 :$ 1 : limits; 1 : integrand and constant

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Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

3. A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

   (a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

   (b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

   (c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

   (d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

END OF PART A OF SECTION II

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2003 SCORING GUIDELINES (Form B)

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi\int_{125}^{275}\left(\frac{B(x)}{2}\right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360}\int_0^{360}\frac{B(x)}{2}\,dx$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2}+\frac{B(180)}{2}+\frac{B(300)}{2}\right)\right] =$
$\frac{1}{360}[60\,(30+30+24)] = 14$

$2 : \begin{cases} 1 : B(60)+B(180)+B(300) \\ 1 : \text{answer} \end{cases}$

(c) $\frac{B(x)}{2}$ is the radius, so $\pi\left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

$2 : \begin{cases} 1 : \text{volume in mm}^3 \\ 1 : \text{between } x = 125 \text{ and } x = 275 \end{cases}$

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in (60, 180) and $B'(c_2) = 0$ for some c_2 in (240, 360). The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval $\left(c_1,\, c_2\right)$.

$3 : \begin{cases} 2 : \text{explains why there are two values of } x \text{ where } B'(x) \text{ has the same value} \\ 1 : \text{explains why that means } B''(x) = 0 \text{ for } 0 < x < 360 \end{cases}$

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in (0, 360).

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

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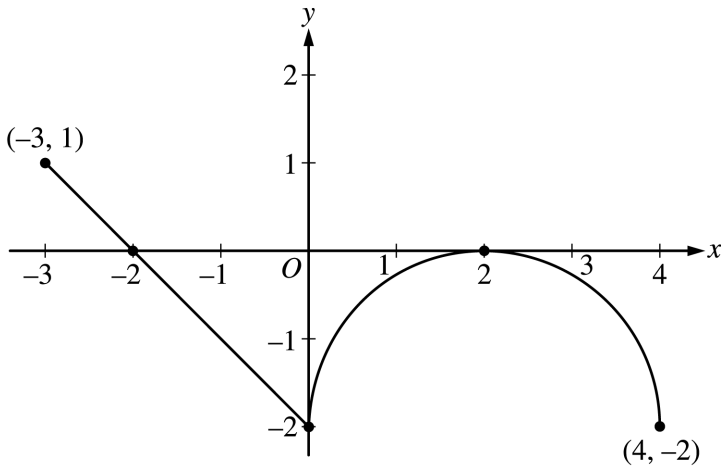

Graph of f'

4. Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.

   (a) On what intervals, if any, is f increasing? Justify your answer.

   (b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

   (c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

   (d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

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Question 4

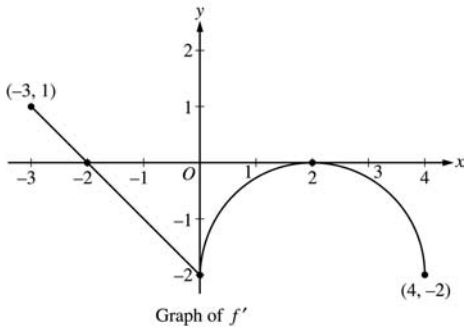
Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $\left(0, 3\right)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.



(a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.

2 : { 1 : interval; 1 : reason }

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

2 : { 1 : $x = 0$ and $x = 2$ only; 1 : justification }

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

1 : equation

(d) $f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt$

$= \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$

$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$

$f(4) - f(0) = \int_{0}^{4} f'(t)\,dt$

$= -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$

$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$

4 : { 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles); 1 : answer for $f(-3)$ using FTC; 1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle − area of semicircle); 1 : answer for $f(4)$ using FTC }

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

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4. A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t \to \infty} \frac{dy}{dx}$.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

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2003 SCORING GUIDELINES (Form B)

Question 4

A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t\to\infty} \frac{dy}{dx}$.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

(a) $x'(t) = 6e^{3t} - 7e^{-7t}$
$y'(t) = 9e^{3t} + 2e^{-2t}$
Velocity vector is $< 6e^{3t} - 7e^{-7t},\ 9e^{3t} + 2e^{-2t} >$

$\text{Speed} = \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{(-1)^2 + 11^2}$
$= \sqrt{122}$

$3 : \begin{cases} 1 : x'(t) \\ 1 : y'(t) \\ 1 : \text{speed} \end{cases}$

(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$

$\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} = \frac{9}{6} = \frac{3}{2}$

$2 : \begin{cases} 1 : \frac{dy}{dx} \text{ in terms of } t \\ 1 : \text{limit} \end{cases}$

(c) Need $y'(t) = 0$, but $9e^{3t} + 2e^{-2t} > 0$ for all t, so none exists.

$2 : \begin{cases} 1 : \text{considers } y'(t) = 0 \\ 1 : \text{explains why none exists} \end{cases}$

(d) Need $x'(t) = 0$ and $y'(t) \neq 0$.
$6e^{3t} = 7e^{-7t}$
$e^{10t} = \frac{7}{6}$
$t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$

$2 : \begin{cases} 1 : \text{considers } x'(t) = 0 \\ 1 : \text{solution} \end{cases}$

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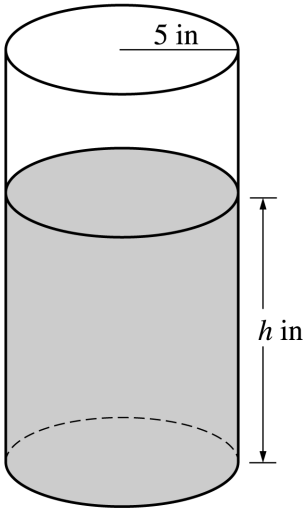

5. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

   (a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

   (b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

   (c) At what time t is the coffeepot empty?

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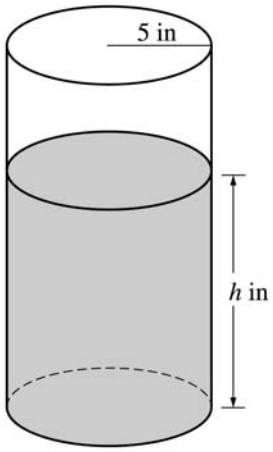
Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$\dfrac{dV}{dt} = 25\pi \dfrac{dh}{dt} = -5\pi\sqrt{h}$

$\dfrac{dh}{dt} = \dfrac{-5\pi\sqrt{h}}{25\pi} = -\dfrac{\sqrt{h}}{5}$

$3 : \begin{cases} 1 : \dfrac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \dfrac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$

(b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$

$\dfrac{1}{\sqrt{h}}\,dh = -\dfrac{1}{5}\,dt$

$2\sqrt{h} = -\dfrac{1}{5}t + C$

$2\sqrt{17} = 0 + C$

$h = \left(-\dfrac{1}{10}t + \sqrt{17}\right)^2$

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \text{ when } t = 0 \\ 1 : \text{solves for } h \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\dfrac{1}{10}t + \sqrt{17}\right)^2 = 0$

$t = 10\sqrt{17}$

1 : answer

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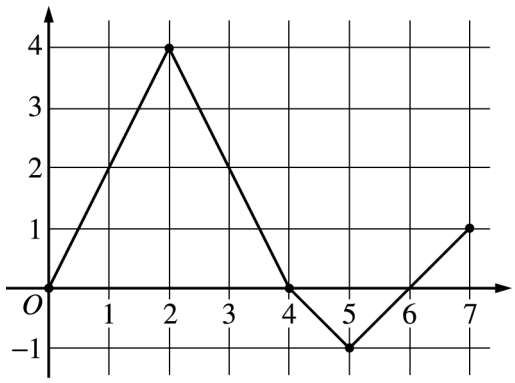
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f

5. Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

   (a) Find $g(3)$, $g'(3)$, and $g''(3)$.

   (b) Find the average rate of change of g on the interval $0 \le x \le 3$.

   (c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b) ? Explain your reasoning.

   (d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

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Question 5

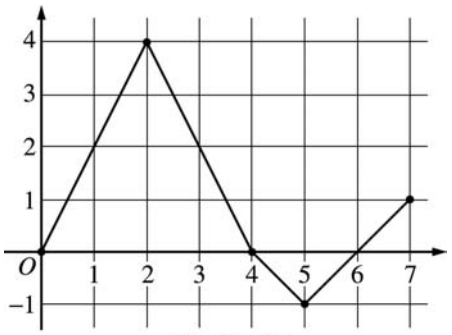
Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.



Graph of f

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad (\text{ignore discussion at } x = 4) \end{cases}$

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6. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

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Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n+1)\,!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n+1)\,!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

(a) $f'(0) =$ coefficient of x term $= 0$

$f''(0) = 2$ (coefficient of x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

4 : { 1 : $f'(0)$; 1 : $f''(0)$; 1 : critical point answer; 1 : reason }

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1) - \left(1 - \frac{1}{3!}\right)\right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

1 : error bound $< \frac{1}{100}$

(c) $y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \cdots$

$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \cdots$

$xy' + y = 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots$

$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$

$= \cos x$

4 : { 1 : series for y'; 1 : series for xy'; 1 : series for $xy' + y$; 1 : identifies series as $\cos x$ }

OR

OR

$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$

$= \sin x$

$xy' + y = (xy)' = (\sin x)' = \cos x$

4 : { 1 : series for $xf(x)$; 1 : identifies series as $\sin x$; 1 : handles $xy' + y$; 1 : makes connection }

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6. The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

   (a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

   (b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

   (c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

   (d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

END OF EXAMINATION

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Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1\,;\ f'(2) = \dfrac{2!}{3}\,;\ f''(2) = \dfrac{3!}{3^2}\,;\ f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!\,3^2}(x-2)^2 + \frac{4!}{3!\,3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!\,3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 : { 1 : coefficients $\dfrac{f^{(n)}(2)}{n!}$ in first four terms; 1 : powers of $(x-2)$ in first four terms; 1 : general term }

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \dfrac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 : { 1 : sets up ratio; 1 : limit; 1 : applies ratio test to conclude radius of convergence is 3 }

(c) $g(2) = 3\,;\ g'(2) = f(2)\,;\ g''(2) = f'(2)\,;\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 : { 1 : first four terms; 1 : general term }

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

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Question 1

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(a) $\int_0^{30} F(t)\,dt = 2474$ cars	3 : 1 : limits; 1 : integrand; 1 : answer
(b) $F'(7) = -1.872$ or -1.873 Since $F'(7) < 0$, the traffic flow is decreasing at $t = 7$.	1 : answer with reason
(c) $\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899$ cars/min	3 : 1 : limits; 1 : integrand; 1 : answer
(d) $\frac{F(15) - F(10)}{15 - 10} = 1.517$ or 1.518 cars/min^2	1 : answer
Units of cars/min in (c) and cars/min^2 in (d)	1 : units in (c) and (d)

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CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

(d) Find the x-coordinate of the position of the particle at time $t = 3$.

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AP® CALCULUS BC
2004 SCORING GUIDELINES (Form B)

Question 1

A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.
(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.
(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.
(d) Find the x-coordinate of the position of the particle at time $t = 3$.

(a) At time $t = 0$:

Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{3^2 + 7^2} = \sqrt{58}$

Acceleration vector $= \langle x''(0), y''(0) \rangle = \langle 0, -3 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration vector} \end{cases}$

(b) $\frac{dy}{dx} = \frac{y'(0)}{x'(0)} = \frac{7}{3}$

Tangent line is $y = \frac{7}{3}(x - 4) + 1$

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \end{cases}$

(c) Distance $= \int_0^3 \sqrt{\left(\sqrt{t^4 + 9}\right)^2 + \left(2e^t + 5e^{-t}\right)^2}\, dt$
$= 45.226$ or 45.227

$3 : \begin{cases} 2 : \text{distance integral} \\ \quad \langle -1 \rangle \text{ each integrand error} \\ \quad \langle -1 \rangle \text{ error in limits} \\ 1 : \text{answer} \end{cases}$

(d) $x(3) = 4 + \int_0^3 \sqrt{t^4 + 9}\, dt$
$= 17.930$ or 17.931

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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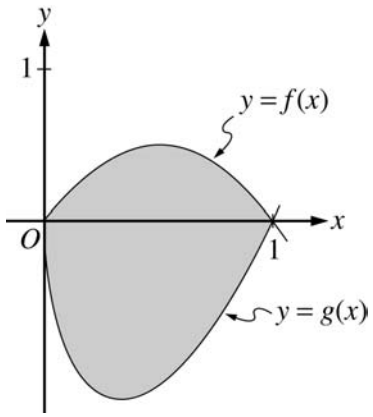

2. Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.

   (a) Find the area of the shaded region enclosed by the graphs of f and g.

   (b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

   (c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

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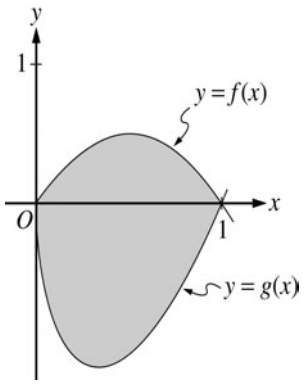
Question 2

Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.



(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

(a) Area $= \int_0^1 (f(x) - g(x))\,dx$

$= \int_0^1 \left(2x(1-x) - 3(x-1)\sqrt{x}\right)\,dx = 1.133$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^1 \left((2-g(x))^2 - (2-f(x))^2\right)\,dx$

$= \pi\int_0^1 \left(\left(2-3(x-1)\sqrt{x}\right)^2 - (2-2x(1-x))^2\right)\,dx$

$= 16.179$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ \quad \langle -1 \rangle \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if integral not of form } c\int_a^b \left(R^2(x) - r^2(x)\right)\,dx \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^1 (h(x) - g(x))^2\,dx$

$\int_0^1 \left(kx(1-x) - 3(x-1)\sqrt{x}\right)^2\,dx = 15$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

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CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

2. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by

$$T(x) = 7 - 9(x - 2)^2 - 3(x - 2)^3.$$

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$?

If not, explain why not.

If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$?

If not, explain why not.

If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

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Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

(a) $f(2) = T(2) = 7$
$\frac{f''(2)}{2!} = -9$ so $f''(2) = -18$

$2 : \begin{cases} 1 : f(2) = 7 \\ 1 : f''(2) = -18 \end{cases}$

(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$.
Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.

$2 : \begin{cases} 1 : \text{states } f'(2) = 0 \\ 1 : \text{declares } f(2) \text{ as a relative maximum because } f''(2) < 0 \end{cases}$

(c) $f(0) \approx T(0) = -5$
It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.

$3 : \begin{cases} 1 : f(0) \approx T(0) = -5 \\ 1 : \text{declares that it is not possible to determine} \\ 1 : \text{reason} \end{cases}$

(d) Lagrange error bound $= \frac{6}{4!}|0-2|^4 = 4$
$f(0) \le T(0) + 4 = -1$
Therefore, $f(0)$ is negative.

$2 : \begin{cases} 1 : \text{value of Lagrange error bound} \\ 1 : \text{explanation} \end{cases}$

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3. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 3 + \cos(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

   (a) Find the x-coordinate of the position of the object at time $t = 4$.

   (b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

   (c) Find the speed of the object at time $t = 2$.

   (d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

END OF PART A OF SECTION II

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Question 3

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 3 + \cos\left(t^2\right)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the x-coordinate of the position of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

(a) $x(4) = x(2) + \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt$

$= 1 + \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt = 7.132$ or 7.133

$3 : \begin{cases} 1 : \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt \\ 1 : \text{handles initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2} = \frac{-7}{3 + \cos 4} = -2.983$

$y - 8 = -2.983(x - 1)$

$2 : \begin{cases} 1 : \text{finds } \left.\frac{dy}{dx}\right|_{t=2} \\ 1 : \text{equation} \end{cases}$

(c) The speed of the object at time $t = 2$ is $\sqrt{(x'(2))^2 + (y'(2))^2} = 7.382$ or 7.383.

$1 : \text{answer}$

(d) $x''(4) = 2.303$

$y'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2t + 1)\left(3 + \cos\left(t^2\right)\right)$

$y''(4) = 24.813$ or 24.814

The acceleration vector at $t = 4$ is $\langle 2.303, 24.813 \rangle$ or $\langle 2.303, 24.814 \rangle$.

$3 : \begin{cases} 1 : x''(4) \\ 1 : \frac{dy}{dt} \\ 1 : \text{answer} \end{cases}$

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t (minutes)	0	5	10	15	20	25	30	35	40
$v(t)$ (miles per minute)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

3. A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

   (a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

   (b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

   (c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure.

   (d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

END OF PART A OF SECTION II

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Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

(a) Midpoint Riemann sum is
$10 \cdot [v(5) + v(15) + v(25) + v(35)]$
$= 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229$
The integral gives the total distance in miles that the plane flies during the 40 minutes.

3 : { 1 : $v(5) + v(15) + v(25) + v(35)$; 1 : answer; 1 : meaning with units }

(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.

2 : { 1 : two instances; 1 : justification }

(c) $f'(23) = -0.407$ or -0.408 miles per minute^2

1 : answer with units

(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$
$= 5.916$ miles per minute

3 : { 1 : limits; 1 : integrand; 1 : answer }

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

   (a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

   (b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

   (c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P ? Justify your answer.

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Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a) $2x + 8yy' = 3y + 3xy'$

$(8y - 3x)y' = 3y - 2x$

$y' = \frac{3y - 2x}{8y - 3x}$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3,\ 3y = 6$

$y = 2$

$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3 : \begin{cases} 1 : \frac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$

At $P = (3, 2),\ \frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

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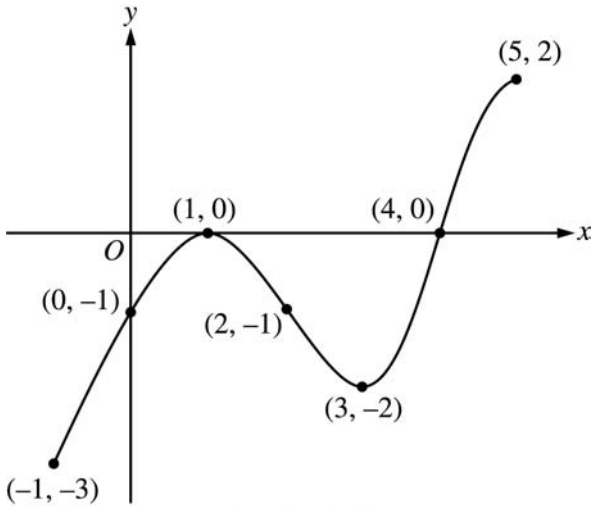

Graph of f'

4. The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

   (a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

   (b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

   (c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

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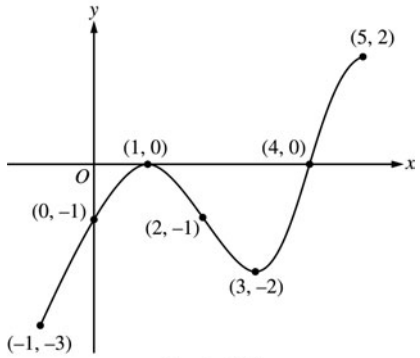
Question 4

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.



Graph of f'

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

(b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

(a) $x = 1$ and $x = 3$ because the graph of f' changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$.

$2 : \begin{cases} 1 : x = 1, x = 3 \\ 1 : \text{reason} \end{cases}$

(b) The function f decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$.
Therefore, the absolute minimum value for f is at $x = 4$.
The absolute maximum value must occur at $x = -1$ or at $x = 5$.

$$f(5) - f(-1) = \int_{-1}^{5} f'(t)\,dt < 0$$

Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$.

$4 : \begin{cases} 1 : \text{indicates } f \text{ decreases then increases} \\ 1 : \text{eliminates } x = 5 \text{ for maximum} \\ 1 : \text{absolute minimum at } x = 4 \\ 1 : \text{absolute maximum at } x = -1 \end{cases}$

(c) $g'(x) = f(x) + xf'(x)$
$g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$
$g(2) = 2f(2) = 12$

Tangent line is $y = 4(x - 2) + 12$

$3 : \begin{cases} 2 : g'(x) \\ 1 : \text{tangent line} \end{cases}$

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

5. A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

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Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12.

If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$.

If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$

(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.

1 : answer

(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$

$\ln|Y| = \frac{t}{5} - \frac{t^2}{120} + C$

$Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$

$K = 3$

$Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(d) $\lim_{t\to\infty} Y(t) = 0$

1 : answer
$0/1$ if Y is not exponential

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5. Let g be the function given by $g(x) = \frac{1}{\sqrt{x}}$.

   (a) Find the average value of g on the closed interval $[1, 4]$.

   (b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

   (c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

   (d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b \to \infty} \left[\frac{\int_a^b f(x)\,dx}{b - a} \right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

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Question 5

Let g be the function given by $g(x) = \frac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b\to\infty}\left[\frac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

(a) $\frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big|_1^4 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

2 : { 1 : integral; 1 : antidifferentiation and evaluation }

(b) Volume $= \pi\int_1^4 \frac{1}{x}\,dx = \pi \ln x\Big|_1^4 = \pi \ln 4$

2 : { 1 : integral; 1 : antidifferentiation and evaluation }

(c) The cross section at x has area $\pi\left(\frac{1}{\sqrt{x}}\right)^2 = \frac{\pi}{x}$

Average value $= \frac{1}{3}\int_1^4 \frac{\pi}{x}\,dx = \frac{1}{3}\pi \ln 4$

1 : answer

(d) $\int_4^\infty g(x)\,dx = \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 4\right) = \infty$

This limit is not finite, so the integral is divergent.

$\frac{\int_4^b g(x)\,dx}{b-4} = \frac{1}{b-4}\int_4^b \frac{1}{\sqrt{x}}\,dx = \frac{2\sqrt{b}-4}{b-4}$

$\lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$

4 : { 1 : $\int_4^b g(x)\,dx = 2\sqrt{b} - 4$; 1 : indicates integral diverges; 1 : $\frac{1}{b-4}\int_4^b g(x)\,dx = \frac{2\sqrt{b}-4}{b-4}$; 1 : finite limit as $b \to \infty$ }

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6. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

   (a) Find $P(x)$.

   (b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

   (c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$.

   (d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

END OF EXAMINATION

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Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$

$f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$

$f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$

$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$

4 : $P(x)$

$\langle -1 \rangle$ each error or missing term

deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error

deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$

2 : { 1 : magnitude; 1 : sign }

(c) $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$

$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$

1 : error bound in an appropriate inequality

(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt$

$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$

2 : third-degree Taylor polynomial for G about $x = 0$

$\langle -1 \rangle$ each incorrect or missing term

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

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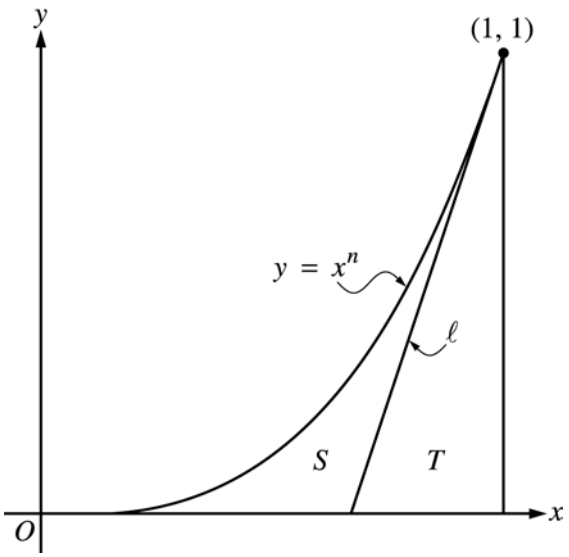

6. Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

   (a) Find $\int_0^1 x^n \, dx$ in terms of n.

   (b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

   (c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

END OF EXAMINATION

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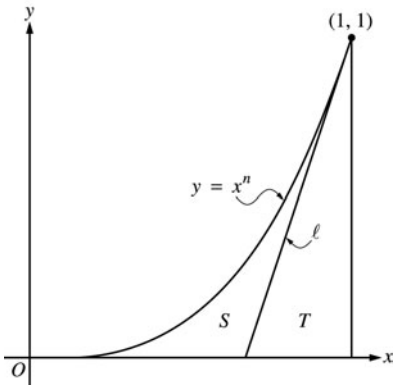
Question 6

Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n \, dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.



(a) $\int_0^1 x^n \, dx = \left. \frac{x^{n+1}}{n+1} \right|_0^1 = \frac{1}{n+1}$

$2 : \begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T.

$\frac{1}{b}$ is the slope of line ℓ, which is n

$\text{Area}(T) = \frac{1}{2} b(1) = \frac{1}{2n}$

$3 : \begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \frac{1}{n} \\ 1 : \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \int_0^1 x^n \, dx - \text{Area}(T)$

$= \frac{1}{n+1} - \frac{1}{2n}$

$\frac{d}{dn} \text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

$2n^2 = (n+1)^2$

$\sqrt{2}\, n = (n+1)$

$n = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2}$

$4 : \begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ 1 : \text{derivative} \\ 1 : \text{sets derivative equal to 0} \\ 1 : \text{solves for } n \end{cases}$

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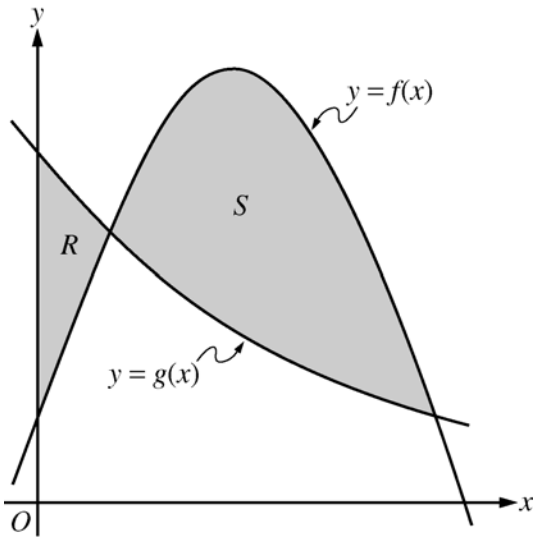
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the area of S.
   (c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.

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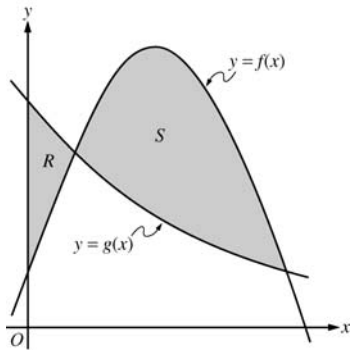
Question 1

Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.

$f(x) = g(x)$ when $\frac{1}{4} + \sin(\pi x) = 4^{-x}$.

f and g intersect when $x = 0.178218$ and when $x = 1$.
Let $a = 0.178218$.

(a) $\int_0^a (g(x) - f(x))\,dx = 0.064$ or 0.065

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) $\int_a^1 (f(x) - g(x))\,dx = 0.410$

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $\pi\int_a^1 \left((f(x)+1)^2 - (g(x)+1)^2\right)dx = 4.558$ or 4.559

3 : { 2 : integrand; 1 : limits, constant, and answer }

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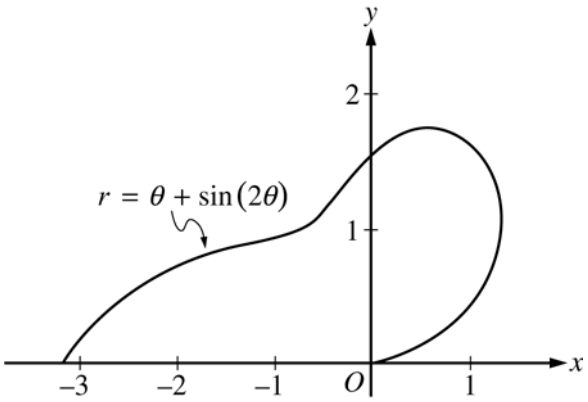

2. The curve above is drawn in the *xy*-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

   (a) Find the area bounded by the curve and the *x*-axis.

   (b) Find the angle θ that corresponds to the point on the curve with *x*-coordinate -2.

   (c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

   (d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

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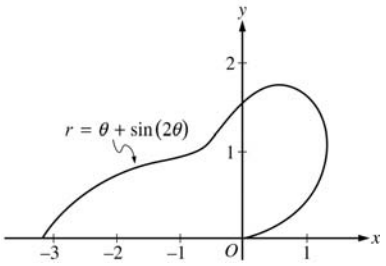
Question 2

The curve above is drawn in the *xy*-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



(a) Find the area bounded by the curve and the *x*-axis.

(b) Find the angle θ that corresponds to the point on the curve with *x*-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) $\text{Area} = \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = 4.382$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$

$\theta = 2.786$

$2 : \begin{cases} 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$2 : \begin{cases} 1 : \text{information about } r \\ 1 : \text{information about the curve} \end{cases}$

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\theta = \frac{\pi}{3}$.

$2 : \begin{cases} 1 : \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1 : \text{answer with justification} \end{cases}$

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Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

3. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

   (a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

   (b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

   (c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

   (d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

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END OF PART A OF SECTION II

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Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8) - T(6)}{8 - 6} = \dfrac{55 - 62}{2} = -\dfrac{7}{2}$ °C/cm

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$A = \dfrac{100 + 93}{2}\cdot 1 + \dfrac{93 + 70}{2}\cdot 4 + \dfrac{70 + 62}{2}\cdot 1 + \dfrac{62 + 55}{2}\cdot 2$

Average temperature $\approx \dfrac{1}{8}A = 75.6875$°C

3 : 1 : $\dfrac{1}{8}\int_0^8 T(x)\,dx$; 1 : trapezoidal sum; 1 : answer

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$°C

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : 1 : value; 1 : meaning

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70 - 93}{5 - 1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62 - 70}{6 - 5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

2 : 1 : two slopes of secant lines; 1 : answer with explanation

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

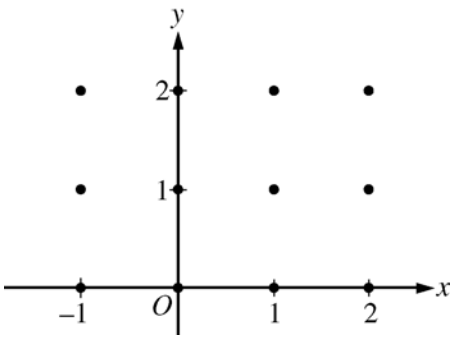
No calculator is allowed for these problems.

---

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$.

   (**Note: Use the axes provided in the pink test booklet.**)



   (b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

   (c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

   (d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

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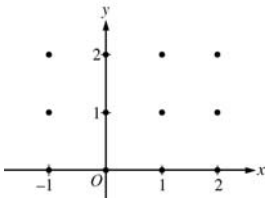
Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.



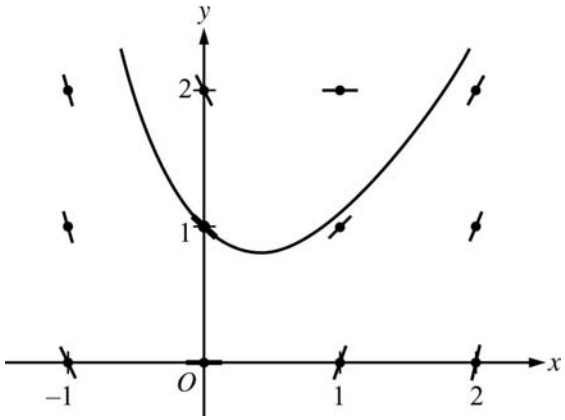
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$. (**Note: Use the axes provided in the pink test booklet.**)

(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

(a)



$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{curve through } (0, 1) \end{cases}$

(b) $\frac{dy}{dx} = 0$ when $2x = y$

The y-coordinate is $2\ln\left(\frac{3}{2}\right)$.

$2 : \begin{cases} 1 : \text{sets } \frac{dy}{dx} = 0 \\ 1 : \text{answer} \end{cases}$

(c) $f(-0.2) \approx f(0) + f'(0)(-0.2)$
$= 1 + (-1)(-0.2) = 1.2$
$f(-0.4) \approx f(-0.2) + f'(-0.2)(-0.2)$
$\approx 1.2 + (-1.6)(-0.2) = 1.52$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{Euler approximation to } f(-0.4) \end{cases}$

(d) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - 2x + y$

$\frac{d^2y}{dx^2}$ is positive in quadrant II because $x < 0$ and $y > 0$.

$1.52 < f(-0.4)$ since all solution curves in quadrant II are concave up.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{answer with reason} \end{cases}$

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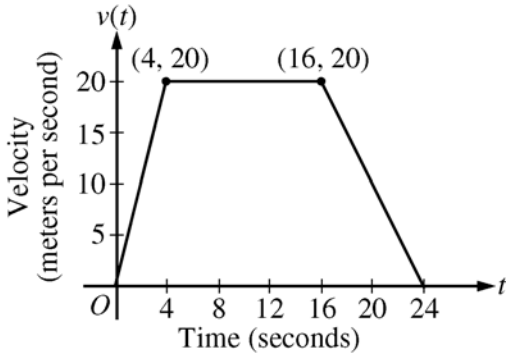

5. A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

   (a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

   (b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

   (c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

   (d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

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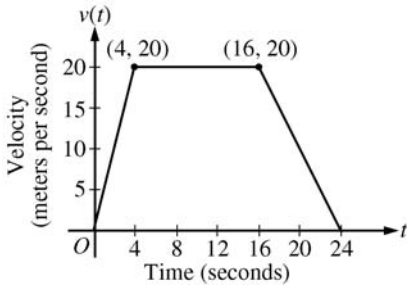
Question 5

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.



(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$

The car travels 360 meters in these 24 seconds.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning with units} \end{cases}$

(b) $v'(4)$ does not exist because

$\lim_{t \to 4^-}\left(\frac{v(t) - v(4)}{t - 4}\right) = 5 \ne 0 = \lim_{t \to 4^+}\left(\frac{v(t) - v(4)}{t - 4}\right)$.

$v'(20) = \frac{20 - 0}{16 - 24} = -\frac{5}{2}\ \text{m/sec}^2$

$3 : \begin{cases} 1 : v'(4) \text{ does not exist, with explanation} \\ 1 : v'(20) \\ 1 : \text{units} \end{cases}$

(c) $a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$2 : \begin{cases} 1 : \text{finds the values } 5,\ 0,\ -\frac{5}{2} \\ 1 : \text{identifies constants with correct intervals} \end{cases}$

(d) The average rate of change of v on $[8, 20]$ is $\frac{v(20) - v(8)}{20 - 8} = -\frac{5}{6}\ \text{m/sec}^2$.

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

$2 : \begin{cases} 1 : \text{average rate of change of } v \text{ on } [8, 20] \\ 1 : \text{answer with explanation} \end{cases}$

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6. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

   (a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

   (b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

   (c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

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END OF EXAM

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Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \dfrac{1!}{3^2}\cdot\dfrac{1}{2!}(x-2)^2 + \dfrac{3!}{3^4}\cdot\dfrac{1}{4!}(x-2)^4 + \dfrac{5!}{3^6}\cdot\dfrac{1}{6!}(x-2)^6$

3 : { 1 : polynomial about $x = 2$; 2 : $P_6(x)$; $\langle -1 \rangle$ each incorrect term; $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality }

(b) $\dfrac{(2n-1)!}{3^{2n}}\cdot\dfrac{1}{(2n)!} = \dfrac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$f(x) = 7 + \displaystyle\sum_{n=1}^{\infty}\frac{1}{2n\cdot 3^{2n}}(x-2)^{2n}$.

$$L = \lim_{n\to\infty}\left|\frac{\dfrac{1}{2(n+1)}\cdot\dfrac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\dfrac{1}{2n}\cdot\dfrac{1}{3^{2n}}(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^2 3^{2n}}(x-2)^2\right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \displaystyle\sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

5 : { 1 : sets up ratio; 1: computes limit of ratio; 1: identifies interior of interval of convergence; 1 : considers both endpoints; 1 : analysis/conclusion for both endpoints }

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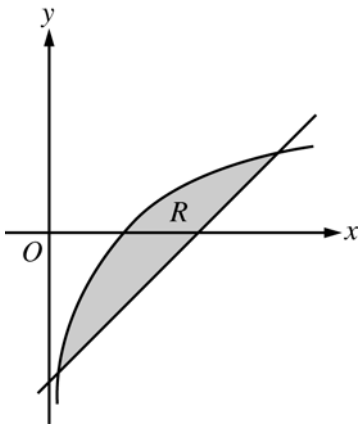
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.

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AP® CALCULUS AB
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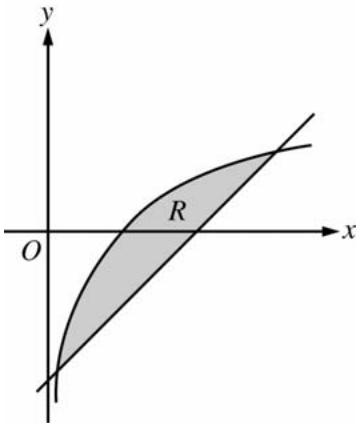
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$

3 : { 1 : integrand; 1 : limits; 1 : answer }

(b) Volume $= \pi \int_S^T \left((\ln(x) + 3)^2 - (x - 2 + 3)^2 \right) dx$
$= 34.198$ or 34.199

3 : { 2 : integrand; 1 : limits, constant, and answer }

(c) Volume $= \pi \int_{S-2}^{T-2} \left((y+2)^2 - \left(e^y\right)^2 \right) dy$

3 : { 2 : integrand; 1 : limits and constant }

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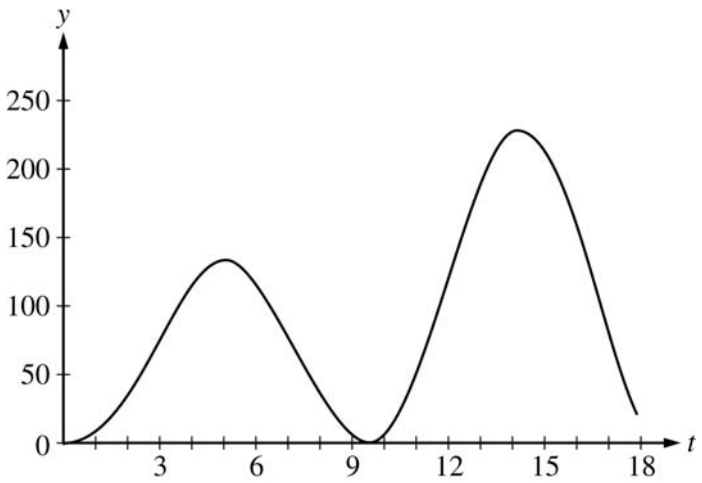

2. At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.

   (a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

   (b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

   (c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

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GO ON TO THE NEXT PAGE.

AP® CALCULUS AB
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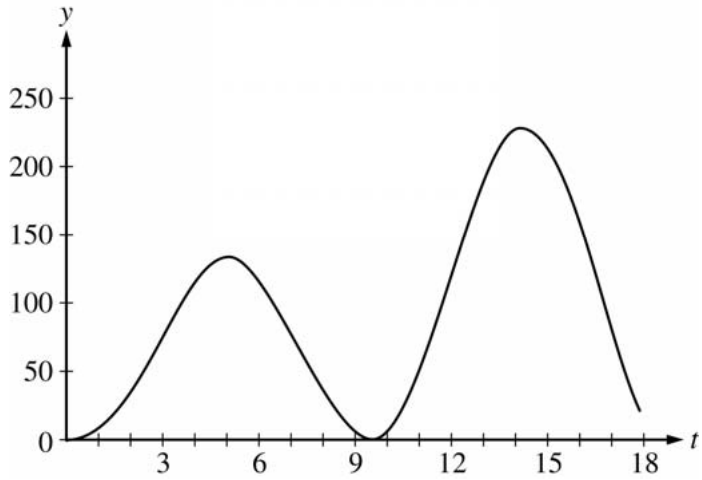
Question 2

At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.



(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(a) $\int_0^{18} L(t)\,dt \approx 1658$ cars

2 : { 1 : setup; 1 : answer }

(b) $L(t) = 150$ when $t = 12.42831,\ 16.12166$
Let $R = 12.42831$ and $S = 16.12166$
$L(t) \ge 150$ for t in the interval $[R, S]$
$\frac{1}{S-R}\int_R^S L(t)\,dt = 199.426$ cars per hour

3 : { 1 : t-interval when $L(t) \ge 150$; 1 : average value integral; 1 : answer with units }

(c) For the product to exceed 200,000, the number of cars turning left in a two-hour interval must be greater than 400.

$\int_{13}^{15} L(t)\,dt = 431.931 > 400$

4 : { 1 : considers 400 cars; 1 : valid interval $[h, h+2]$; 1 : value of $\int_h^{h+2} L(t)\,dt$; 1 : answer and explanation }

OR

The number of cars turning left will be greater than 400 on a two-hour interval if $L(t) \ge 200$ on that interval.
$L(t) \ge 200$ on any two-hour subinterval of $[13.25304, 15.32386]$.

4 : { 1 : considers 200 cars per hour; 1 : solves $L(t) \ge 200$; 1 : discusses 2 hour interval; 1 : answer and explanation }

Yes, a traffic signal is required.

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3. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1+t^3}$$

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t\to\infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

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END OF PART A OF SECTION II

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Question 3

An object moving along a curve in the *xy*-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1+t^3}$$

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t\to\infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

(a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740 \rangle$

Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207$ or 1.208

$2 : \begin{cases} 1 : \text{acceleration} \\ 1 : \text{speed} \end{cases}$

(b) $\sin^{-1}\left(1 - 2e^{-t}\right) = 0$

$1 - 2e^{-t} = 0$

$t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$

$2 : \begin{cases} 1 : x'(t) = 0 \\ 1 : \text{answer} \end{cases}$

(c) $m(t) = \dfrac{4t}{1+t^3} \cdot \dfrac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}$

$\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(\dfrac{4t}{1+t^3} \cdot \dfrac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}\right)$

$= 0\left(\dfrac{1}{\sin^{-1}(1)}\right) = 0$

$2 : \begin{cases} 1 : m(t) \\ 1 : \text{limit value} \end{cases}$

(d) Since $\lim_{t\to\infty} x(t) = \infty$,

$c = \lim_{t\to\infty} y(t) = -3 + \int_2^\infty \dfrac{4t}{1+t^3}\, dt$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{initial value consistent with lower limit} \end{cases}$

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

4. Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

   (a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

   (c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

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Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20} \text{ ft/sec}^2$$

1 : answer

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is
$20[v(20) + v(40) + v(60)]$
$= 20[22 + 35 + 44] = 2020$ ft

3 : { 1 : explanation; 1 : uses $v(20)$, $v(40)$, $v(60)$; 1 : value }

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$
$2 = v_B(0) = 6 + C$
$v_B(t) = 6\sqrt{t+1} - 4$
$v_B(80) = 50 > 49 = v(80)$

Rocket B is traveling faster at time $t = 80$ seconds.

4 : { 1 : $6\sqrt{t+1}$; 1 : constant of integration; 1 : uses initial condition; 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion }

Units of ft/sec^2 in (a) and ft in (b)

1 : units in (a) and (b)

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5. Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

   (a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

   (b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

   (c) Find the second-degree Taylor polynomial for f about $x = -1$.

   (d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

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Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(a) $\left.\frac{dy}{dx}\right|_{(-1, -4)} = 6$

$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(-1, -4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$

3 : $\begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(-1, -4)} \\ 1 : \frac{d^2y}{dx^2} \\ 1 : \left.\frac{d^2y}{dx^2}\right|_{(-1, -4)} \end{cases}$

(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right|_{(k, 0)} = 0$.

The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right|_{(k, 0)} = 5k^2 + 3 > 0$ for all k.

2 : $\begin{cases} 1 : \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1 : \text{answer and explanation} \end{cases}$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

2 : $\begin{cases} 1 : \text{quadratic and centered at } x = -1 \\ 1 : \text{coefficients} \end{cases}$

(d) $f(-1) = -4$

$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$

$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$

2 : $\begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation to } f(0) \end{cases}$

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6. The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n\, nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n\, x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

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END OF EXAM

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Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $\left|\dfrac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \dfrac{n+1}{(-1)^n nx^n}\right| = \dfrac{(n+1)^2}{(n+2)(n)} \cdot |x|$

$\lim\limits_{n\to\infty} \dfrac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\dfrac{1}{2} + \dfrac{2}{3} - \dfrac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

5 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : identifies radius of convergence; 1 : considers both endpoints; 1 : analysis/conclusion for both endpoints }

(b) $f'(x) = -\dfrac{1}{2} + \dfrac{4}{3}x - \dfrac{9}{4}x^2 + \cdots$ and $f'(0) = -\dfrac{1}{2}$.

$g'(x) = -\dfrac{1}{2!} + \dfrac{2}{4!}x - \dfrac{3}{6!}x^2 + \cdots$ and $g'(0) = -\dfrac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \dfrac{4}{3}$ and $g''(0) = \dfrac{2}{4!} = \dfrac{1}{12}$.

Thus, $y''(0) = \dfrac{4}{3} - \dfrac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

4 : { 1 : $y'(0)$; 1 : $y''(0)$; 1 : conclusion; 1 : reasoning }

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1 + x^2}$ and below by the horizontal line $y = 2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the x-axis.

   (c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

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Question 1

Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$

1 : correct limits in an integral in (a), (b), or (c)

(a) Area $= \displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)dx = 37.961$ or 37.962

2 : { 1 : integrand; 1 : answer }

(b) Volume $= \pi\displaystyle\int_{-3}^{3}\left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right)dx = 1871.190$

3 : { 2 : integrand; 1 : answer }

(c) Volume $= \dfrac{\pi}{2}\displaystyle\int_{-3}^{3}\left(\frac{1}{2}\left(\frac{20}{1+x^2} - 2\right)\right)^2 dx$

$= \dfrac{\pi}{8}\displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)^2 dx = 174.268$

3 : { 2 : integrand; 1 : answer }

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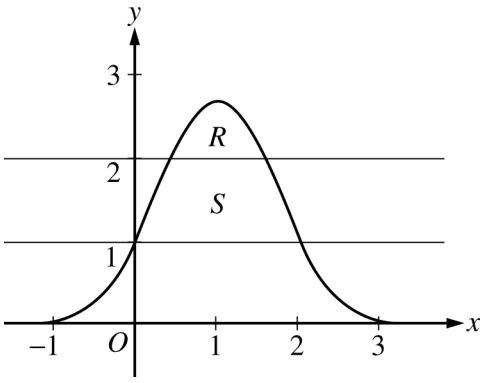
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.
   (a) Find the area of R.
   (b) Find the area of S.
   (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

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2007 SCORING GUIDELINES (Form B)

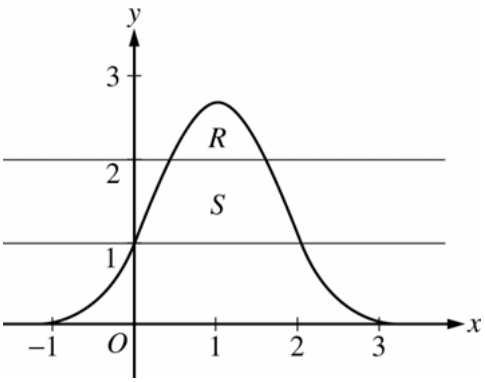
Question 1

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

(b) Find the area of S.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.



$e^{2x-x^2} = 2$ when $x = 0.446057, 1.553943$

Let $P = 0.446057$ and $Q = 1.553943$

(a) Area of $R = \displaystyle\int_P^Q \left(e^{2x-x^2} - 2\right) dx = 0.514$

3 : { 1 : integrand; 1 : limits; 1 : answer }

(b) $e^{2x-x^2} = 1$ when $x = 0,\ 2$

Area of $S = \displaystyle\int_0^2 \left(e^{2x-x^2} - 1\right) dx$ − Area of R

$= 2.06016 -$ Area of $R = 1.546$

OR

$\displaystyle\int_0^P \left(e^{2x-x^2} - 1\right) dx + (Q - P) \cdot 1 + \int_Q^2 \left(e^{2x-x^2} - 1\right) dx$

$= 0.219064 + 1.107886 + 0.219064 = 1.546$

3 : { 1 : integrand; 1 : limits; 1 : answer }

(c) Volume $= \pi \displaystyle\int_P^Q \left(\left(e^{2x-x^2} - 1\right)^2 - (2-1)^2\right) dx$

3 : { 2 : integrand; 1 : constant and limits }

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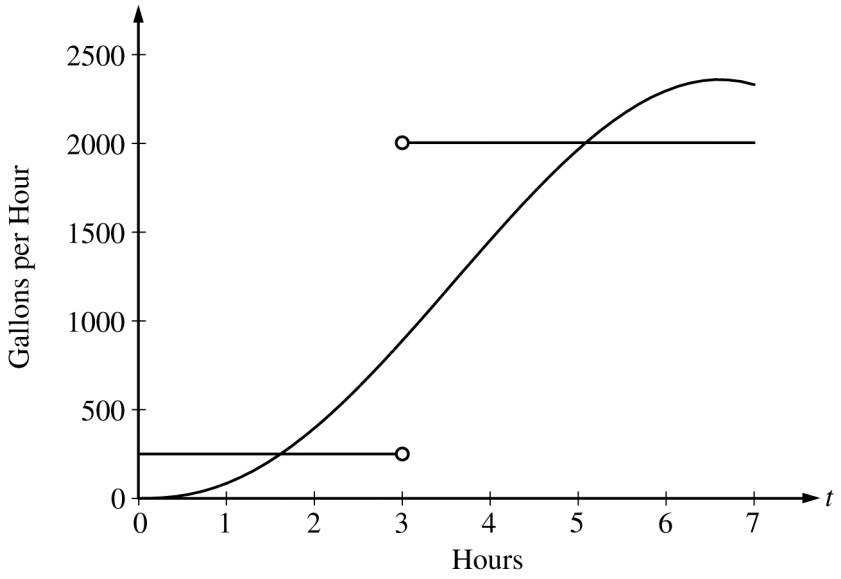

2. The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

   (i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

   (ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$

   The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

   (a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

   (b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

   (c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

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GO ON TO THE NEXT PAGE.

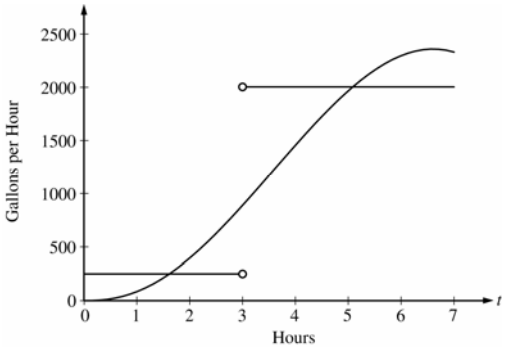
AP® CALCULUS AB
2007 SCORING GUIDELINES

Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is $g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases}$ gallons per hour.



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0$, 3, and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{integrand} \\ 1 : \text{amount of water at } t = 3 \\ 1 : \text{amount of water at } t = 7 \\ 1 : \text{conclusion} \end{cases}$

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2. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

for $t \geq 0$. At time $t = 0$, the object is at position $(-3, -4)$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the speed of the object at time $t = 4$.

(b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

(c) Find $x(4)$.

(d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

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2007 SCORING GUIDELINES (Form B)

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

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(a) Find the speed of the object at time $t = 4$.

(b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

(c) Find $x(4)$.

(d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

(a) Speed $= \sqrt{x'(4)^2 + y'(4)^2} = 2.912$

1 : speed at $t = 4$

(b) Distance $= \displaystyle\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 6.423$

2 : { 1 : integral; 1 : answer }

(c) $x(4) = x(0) + \displaystyle\int_0^4 x'(t)\,dt$

$= -3 + 2.10794 = -0.892$

3 : { 2 : { 1 : integrand; 1 : uses $x(0) = -3$ }; 1 : answer }

(d) The slope is 2, so $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2$, or $\ln\left(t^2 + 1\right) = 2\arctan\left(\frac{t}{1+t}\right)$.

Since $t > 0$, $t = 1.35766$. At this time, the acceleration is $\langle x''(t), y''(t)\rangle\big|_{t=1.35766} = \langle 0.135, 0.955\rangle$.

3 : { 1 : $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2$; 1 : t-value; 1 : values for x'' and y'' }

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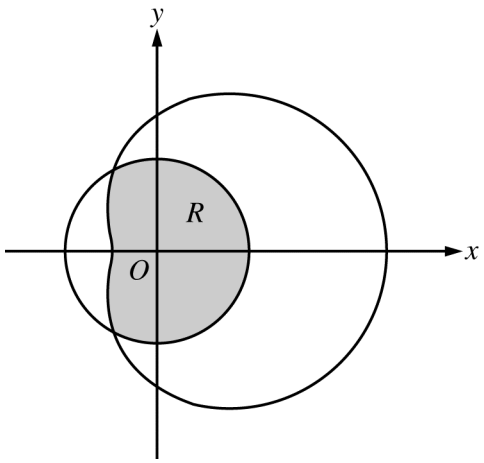

3. The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

   (a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

   (b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

   (c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

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END OF PART A OF SECTION II

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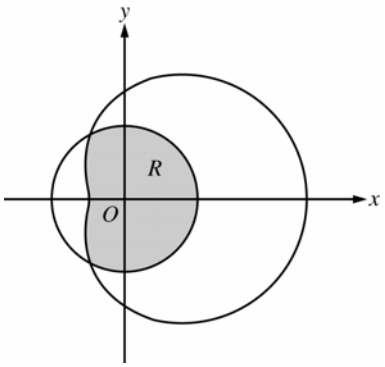
Question 3

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) $\text{Area} = \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\, d\theta$
$= 10.370$

4 : { 1 : area of circular sector; 2 : integral for section of limaçon { 1 : integrand; 1 : limits and constant }; 1 : answer }

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

2 : { 1 : $\left.\frac{dr}{dt}\right|_{\theta=\pi/3}$; 1 : interpretation }

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$
$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

3 : { 1 : expression for y in terms of θ; 1 : $\left.\frac{dy}{dt}\right|_{\theta=\pi/3}$; 1: interpretation }

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3. The wind chill is the temperature, in degrees Fahrenheit $(°F)$, a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is $32°F$, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

   (a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

   (b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

   (c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant $32°F$. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

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END OF PART A OF SECTION II

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2007 SCORING GUIDELINES (Form B)

Question 3

The wind chill is the temperature, in degrees Fahrenheit (°F), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is 32°F, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

(a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

(b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

(c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant 32°F. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

(a) $W'(20) = -22.1 \cdot 0.16 \cdot 20^{-0.84} = -0.285$ or -0.286

When $v = 20$ mph, the wind chill is decreasing at 0.286 °F/mph.

2 : { 1 : value; 1 : explanation }

(b) The average rate of change of W over the interval $5 \le v \le 60$ is $\dfrac{W(60) - W(5)}{60 - 5} = -0.253$ or -0.254.

$W'(v) = \dfrac{W(60) - W(5)}{60 - 5}$ when $v = 23.011$.

3 : { 1 : average rate of change; 1 : $W'(v)$ = average rate of change; 1 : value of v }

(c) $\left.\dfrac{dW}{dt}\right|_{t=3} = \left(\dfrac{dW}{dv} \cdot \dfrac{dv}{dt}\right)\Big|_{t=3} = W'(35) \cdot 5 = -0.892$ °F/hr

OR

$W = 55.6 - 22.1(20 + 5t)^{0.16}$

$\left.\dfrac{dW}{dt}\right|_{t=3} = -0.892$ °F/hr

3 : { 1 : $\dfrac{dv}{dt} = 5$; 1 : uses $v(3) = 35$, or uses $v(t) = 20 + 5t$; 1 : answer }

Units of °F/mph in (a) and °F/hr in (c)

1 : units in (a) and (c)

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

   (a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

   (b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

   (c) Use antidifferentiation to find $f(x)$.

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Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

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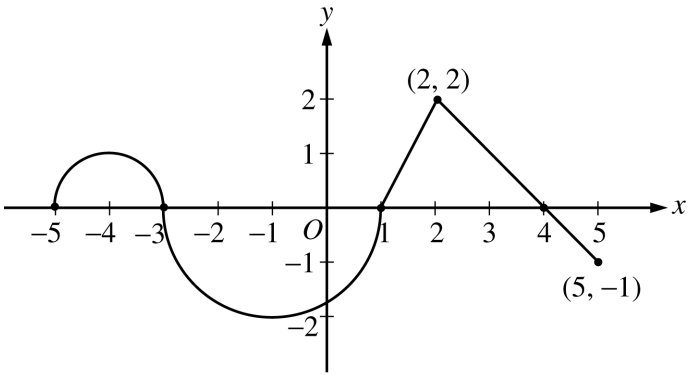
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f'

4. Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.
   (a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.
   (b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.
   (c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.
   (d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

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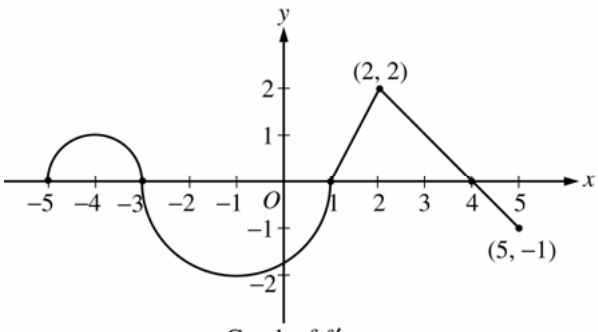
2007 SCORING GUIDELINES (Form B)

Question 4

Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.



Graph of f'

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

(a) $f'(x) = 0$ at $x = -3,\ 1,\ 4$
f' changes from positive to negative at -3 and 4.
Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) f' changes from increasing to decreasing, or vice versa, at $x = -4,\ -1,$ and 2. Thus, the graph of f has points of inflection when $x = -4,\ -1,$ and 2.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{explanation} \end{cases}$

(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$

$f(1) = 3$

$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

$3 : \begin{cases} 1 : \text{identifies } x = 1 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{value and explanation} \end{cases}$

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$.

   (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

   (a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

   (b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

   (c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

   (d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

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Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft
Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{conclusion with reason} \end{cases}$

(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$
$\left.\frac{dV}{dt}\right|_{t=5} = 4\pi(30)^2\, 2 = 7200\pi \text{ ft}^3/\text{min}$

$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$
$= 19.3$ ft
$\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$

(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.

1 : conclusion with reason

Units of ft^3/min in part (b) and ft in part (c)

1 : units in (b) and (c)

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5. Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

   (a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

   (b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

   (c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

   (d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

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AP® CALCULUS BC
2007 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

(a) $\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$

$2 : \begin{cases} 1 : 3 + 2\frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$

(b) If $y = mx + b + e^{rx}$ is a solution, then
$m + re^{rx} = 3x + 2\left(mx + b + e^{rx}\right) + 1$.

If $r \neq 0$: $m = 2b + 1$, $r = 2$, $0 = 3 + 2m$,
so $m = -\frac{3}{2}$, $r = 2$, and $b = -\frac{5}{4}$.

OR

If $r = 0$: $m = 2b + 3$, $r = 0$, $0 = 3 + 2m$,
so $m = -\frac{3}{2}$, $r = 0$, $b = -\frac{9}{4}$.

$3 : \begin{cases} 1 : \frac{dy}{dx} = m + re^{rx} \\ 1 : \text{value for } r \\ 1 : \text{values for } m \text{ and } b \end{cases}$

(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0) \cdot \frac{1}{2} = -2 + (-3) \cdot \frac{1}{2} = -\frac{7}{2}$
$f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$
$f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) \cdot \frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right) \cdot \frac{1}{2} = -\frac{23}{4}$

$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation for } f(1) \end{cases}$

(d) $g'(0) = 3 \cdot 0 + 2 \cdot k + 1 = 2k + 1$
$g(1) \approx g(0) + g'(0) \cdot 1 = k + (2k + 1) = 3k + 1 = 0$
$k = -\frac{1}{3}$

$2 : \begin{cases} 1 : g(0) + g'(0) \cdot 1 \\ 1 : \text{value of } k \end{cases}$

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6. Let f be the function given by $f(x) = e^{-x^2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Use your answer to part (a) to find $\lim_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

   (c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

   (d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

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Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

(a) $e^{-x^2} = 1 + \dfrac{\left(-x^2\right)}{1!} + \dfrac{\left(-x^2\right)^2}{2!} + \dfrac{\left(-x^2\right)^3}{3!} + \cdots + \dfrac{\left(-x^2\right)^n}{n!} + \cdots$

$= 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{6} + \cdots + \dfrac{(-1)^n x^{2n}}{n!} + \cdots$

3 : { 1 : two of $1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6}$; 1 : remaining terms; 1 : general term }

(b) $\dfrac{1 - x^2 - f(x)}{x^4} = -\dfrac{1}{2} + \dfrac{x^2}{6} + \sum_{n=4}^{\infty} \dfrac{(-1)^{n+1} x^{2n-4}}{n!}$

Thus, $\lim_{x \to 0} \left(\dfrac{1 - x^2 - f(x)}{x^4} \right) = -\dfrac{1}{2}$.

1 : answer

(c) $\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \dfrac{t^4}{2} - \dfrac{t^6}{6} + \cdots + \dfrac{(-1)^n t^{2n}}{n!} + \cdots \right) dt$

$= x - \dfrac{x^3}{3} + \dfrac{x^5}{10} - \dfrac{x^7}{42} + \cdots$

Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \dfrac{1}{2} - \left(\dfrac{1}{3}\right)\left(\dfrac{1}{8}\right) = \dfrac{11}{24}$.

3 : { 1 : two terms; 1 : remaining terms; 1 : estimate }

(d) $\left| \int_0^{1/2} e^{-t^2}\,dt - \dfrac{11}{24} \right| < \left(\dfrac{1}{2}\right)^5 \cdot \dfrac{1}{10} = \dfrac{1}{320} < \dfrac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\,dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

2 : { 1 : uses the third term as the error bound; 1 : explanation }

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6. Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

   (a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

   (b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

   (c) The function h satisfies $h(x) = kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

   Find the values of a and k.

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2007 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k\,f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$

$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$

3 : { 1 : two of $6, -2x, \frac{x^2}{3}, -\frac{x^3}{27}$; 1 : remaining terms; 1 : general term }

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so

$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$

$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$

3 : { 1 : two terms; 1 : remaining terms; 1 : general term }

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2k\,e^{-ax/3}$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$

$-2k\,e^{-ax/3} = e^x$

$\frac{-a}{3} = 1$ and $-2k = 1$

$a = -3$ and $k = -\frac{1}{2}$

OR

$f'(x) = -2 + \frac{2}{3}x + \cdots$, so

$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$

$h(x) = 1 + x + \cdots$

$-2k = 1$ and $\frac{2}{3}ak = 1$

$k = -\frac{1}{2}$ and $a = -3$

3 : { 1 : computes $k\,f'(ax)$; 1 : recognizes $h(x) = e^x$, or equates 2 series for $h(x)$; 1 : values for a and k }

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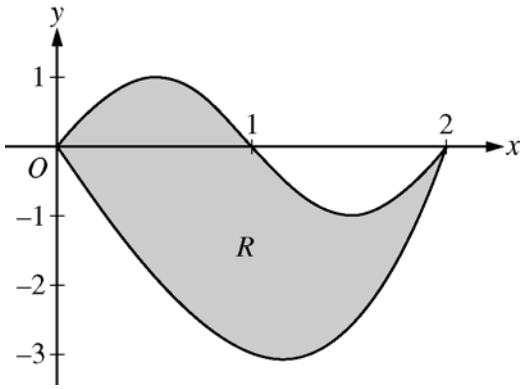
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

   (a) Find the area of R.

   (b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

   (d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

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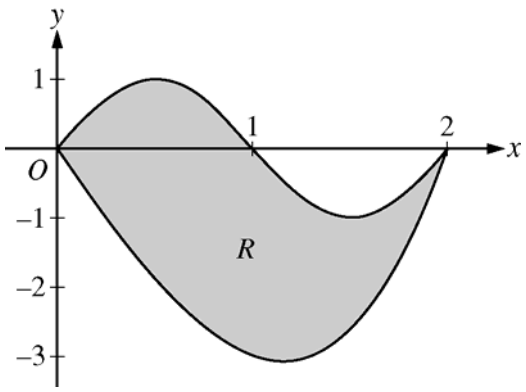
AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$

Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$

2 : { 1 : limits; 1 : integrand }

(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$

2 : { 1 : integrand; 1 : answer }

(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370

2 : { 1 : integrand; 1 : answer }

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CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

1. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

(c) On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5 ?

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

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2008 SCORING GUIDELINES (Form B)

Question 1

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

(c) On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5 ?

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

(a) $a(4) = \langle x''(4), y''(4) \rangle = \langle 0.433, -11.872 \rangle$

1 : answer

(b) $y(0) = 5 + \displaystyle\int_4^0 3\cos\left(\frac{t^2}{2}\right) dt = 1.600$ or 1.601

3 : { 1 : integrand; 1 : uses $y(4) = 5$; 1 : answer }

(c) Speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$

$$= \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)} = 3.5$$

The particle first reaches this speed when $t = 2.225$ or 2.226.

3 : { 1 : expression for speed; 1 : equation; 1 : answer }

(d) $\displaystyle\int_0^4 \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)}\, dt = 13.182$

2 : { 1 : integral; 1 : answer }

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t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2. Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

   (a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

   (b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

   (c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

   (d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

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Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3 : \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3 : \begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[*Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.*]

(d) $\displaystyle\int_0^3 r(t)\,dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

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2. For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

   (a) How many kilometers does the car travel during the first 2 hours?

   (b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

   (c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

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Question 2

For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

(a) How many kilometers does the car travel during the first 2 hours?

(b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

(c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

(a) $\int_0^2 r(t)\,dt = 206.370$ kilometers

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{dg}{dt} = \dfrac{dg}{dx} \cdot \dfrac{dx}{dt}; \quad \dfrac{dx}{dt} = r(t)$

$\left.\dfrac{dg}{dt}\right|_{t=2} = \left.\dfrac{dg}{dx}\right|_{x=206.370} \cdot r(2)$

$= (0.050)(120) = 6$ liters/hour

$3 : \begin{cases} 2 : \text{uses chain rule} \\ 1 : \text{answer with units} \end{cases}$

(c) Let T be the time at which the car's speed reaches 80 kilometers per hour.

Then, $r(T) = 80$ or $T = 0.331453$ hours.

At time T, the car has gone

$x(T) = \int_0^T r(t)\,dt = 10.794097$ kilometers

and has consumed $g(x(T)) = 0.537$ liters of gasoline.

$4 : \begin{cases} 1 : \text{equation } r(t) = 80 \\ 2 : \text{distance integral} \\ 1 : \text{answer} \end{cases}$

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x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

3. Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

   (a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

   (b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

   (c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

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END OF PART A OF SECTION II

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2008 SCORING GUIDELINES

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

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(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$

(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left| h^{(4)}(x) \right| = \frac{584}{9}$.

Therefore, $\left| h(1.9) - P_3(1.9) \right| \le \frac{584}{9} \frac{|1.9-2|^4}{4!}$

$= 2.7037 \times 10^{-4}$

$< 3 \times 10^{-4}$

$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$

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Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

3. A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

   (a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

   (b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

   (c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

   (d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

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END OF PART A OF SECTION II

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2008 SCORING GUIDELINES (Form B)

Question 3

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

(a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

(b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

(c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

(d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

(a) $\frac{(0+7)}{2}\cdot 8 + \frac{(7+8)}{2}\cdot 6 + \frac{(8+2)}{2}\cdot 8 + \frac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$
$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

3 : { 1 : limits and average value constant; 1 : integrand; 1 : answer }

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right)dx = 122.230$ or 122.231 ft^2

2 : { 1 : integra1; 1 : answer }

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is $\frac{1}{20}\int_{40}^{60} C\cdot v(t)\,dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}$.

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

3 : { 1 : volumetric flow integral; 1 : average volumetric flow; 1 : answer with reason }

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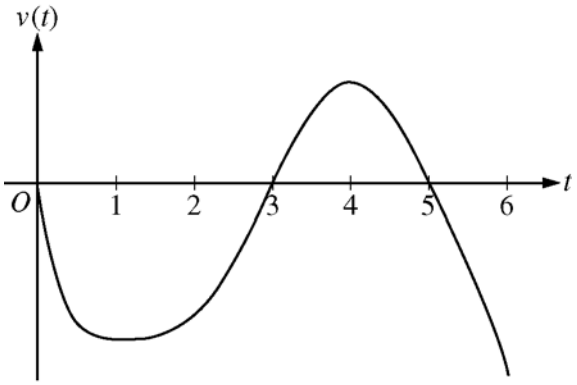
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of v

4. A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

   (a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

   (b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

   (c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

   (d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

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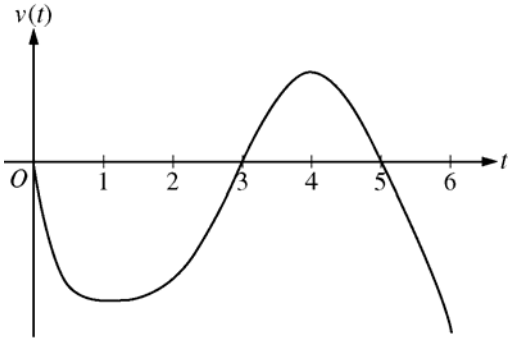
Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.

$$x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$$

$$x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$$

Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.

3 : { 1 : identifies $t = 3$ as a candidate; 1 : considers $\int_0^6 v(t)\,dt$; 1 : conclusion }

(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.

By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.

3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$; 1 : description of motion; 1 : conclusion }

(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.

1 : answer with reason

(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

2 : { 1 : answer; 1 : justification }

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the x-axis.
   (a) Find all values of the constant k for which the area of R equals 2.
   (b) For $k > 0$, write, but do not evaluate, an integral expression in terms of k for the volume of the solid generated when R is rotated about the x-axis.
   (c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.

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Question 4

Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the x-axis.

(a) Find all values of the constant k for which the area of R equals 2.

(b) For $k > 0$, write, but do not evaluate, an integral expression in terms of k for the volume of the solid generated when R is rotated about the x-axis.

(c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.

(a) For $x \geq 0$, $f(x) = x^2(k - x) \geq 0$ if $0 \leq x \leq k$

$$\int_0^k \left(kx^2 - x^3\right) dx = \left(\frac{k}{3}x^3 - \frac{1}{4}x^4\right)\Big|_{x=0}^{x=k} = \frac{k^4}{12}$$

$$\text{Area} = \frac{k^4}{12} = 2; \quad k = \sqrt[4]{24}$$

$4 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{value of integral} \\ 1 : \text{answer} \end{cases}$

(b) $\text{Volume} = \pi\int_0^k \left(kx^2 - x^3\right)^2 dx$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) $\text{Perimeter} = k + \int_0^k \sqrt{1 + \left(2kx - 3x^2\right)^2}\, dx$

$3 : \begin{cases} 1 : \int_0^k \sqrt{1 + (f'(x))^2}\, dx \\ 1 : \text{uses } f'(x) = 2kx - 3x^2 \text{ in integrand} \\ 1 : \text{answer} \end{cases}$

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5. The derivative of a function f is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

   (a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

   (b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

   (c) Find the value of $f(3)$.

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Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2 : \begin{cases} 1: \text{minimum at } x = 3 \\ 1: \text{justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$u = x-3 \quad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$

$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$

$= 7 + 3e - e^3$

$4: \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

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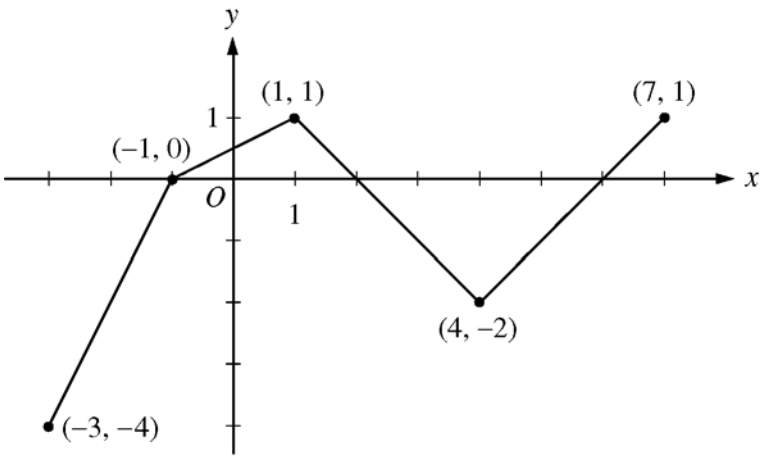

Graph of g'

5. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

   (a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

   (b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

   (c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

   (d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

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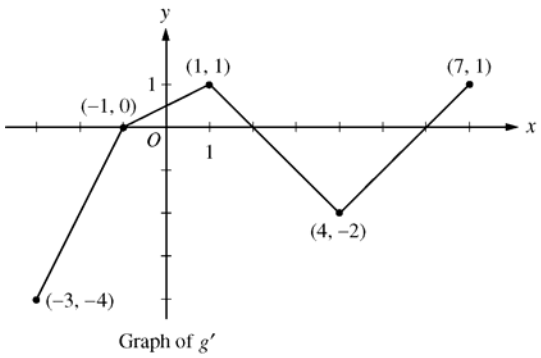
Question 5

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3 : \begin{cases} 1 : \text{identifies } x = 2 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{maximum value and justification} \end{cases}$

(c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2 : \begin{cases} 1 : \text{average value of } g'(x) \\ 1 : \text{answer "No" with reason} \end{cases}$

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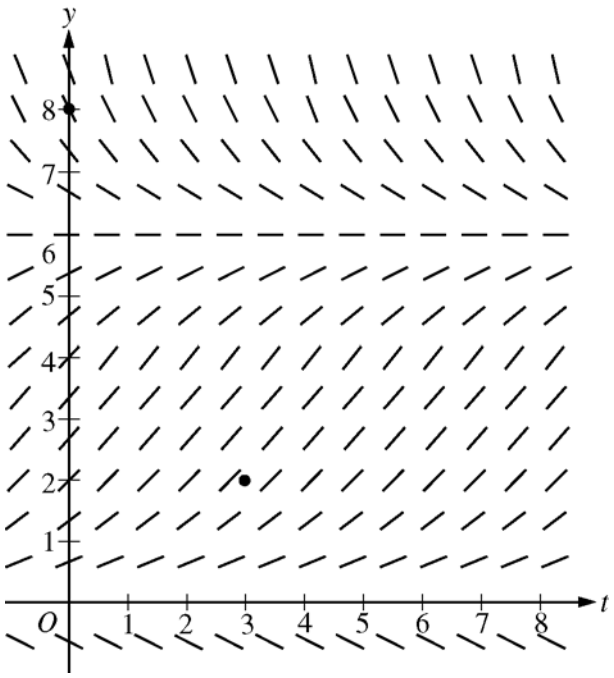
6. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

   (a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.

   (Note: Use the axes provided in the exam booklet.)



   (b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

   (c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

   (d) What is the range of f for $t \geq 0$?

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

END OF EXAM

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2008 SCORING GUIDELINES

Question 6

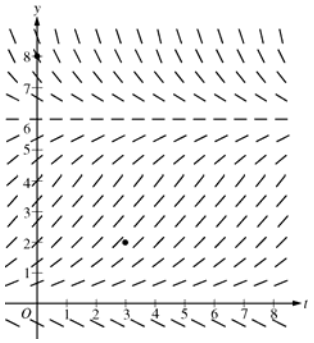
Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

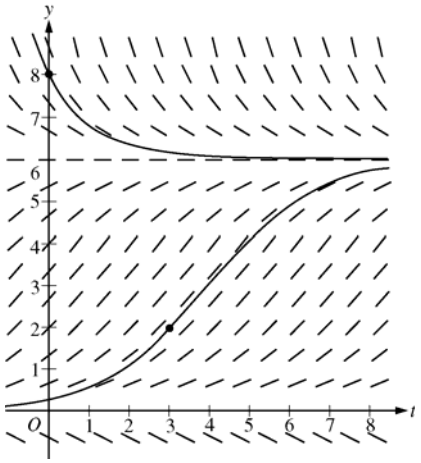
(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?



(a)



$2 : \begin{cases} 1: \text{solution curve through } (0,8) \\ 1: \text{solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8;\ f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2;$ and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

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6. Let f be the function given by $f(x) = \dfrac{2x}{1 + x^2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

   (c) The derivative of $\ln\left(1 + x^2\right)$ is $\dfrac{2x}{1 + x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1 + x^2\right)$ about $x = 0$.

   (d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

WRITE ALL WORK IN THE EXAM BOOKLET.

END OF EXAM

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AP® CALCULUS BC
2008 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

(a) $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$

$\dfrac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$

3 : { 1 : two of the first four terms; 1 : remaining terms; 1 : general term }

(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c) $\ln\left(1+x^2\right) = \displaystyle\int_0^x \frac{2t}{1+t^2}\,dt$

$= \displaystyle\int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$

$= x^2 - \dfrac{1}{2}x^4 + \dfrac{1}{3}x^6 - \dfrac{1}{4}x^8 + \cdots$

2 : { 1 : two of the first four terms; 1 : remaining terms }

(d) $\ln\left(\dfrac{5}{4}\right) = \ln\left(1+\dfrac{1}{4}\right) = \left(\dfrac{1}{2}\right)^2 - \dfrac{1}{2}\left(\dfrac{1}{2}\right)^4 + \dfrac{1}{3}\left(\dfrac{1}{2}\right)^6 - \dfrac{1}{4}\left(\dfrac{1}{2}\right)^8 + \cdots$

Let $A = \left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)^4 = \dfrac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \left|\dfrac{1}{3}\left(\dfrac{1}{2}\right)^6\right| = \dfrac{1}{3}\cdot\dfrac{1}{64} < \dfrac{1}{100}$.

3 : { 1 : uses $x = \dfrac{1}{2}$; 1 : value of A; 1 : justification }

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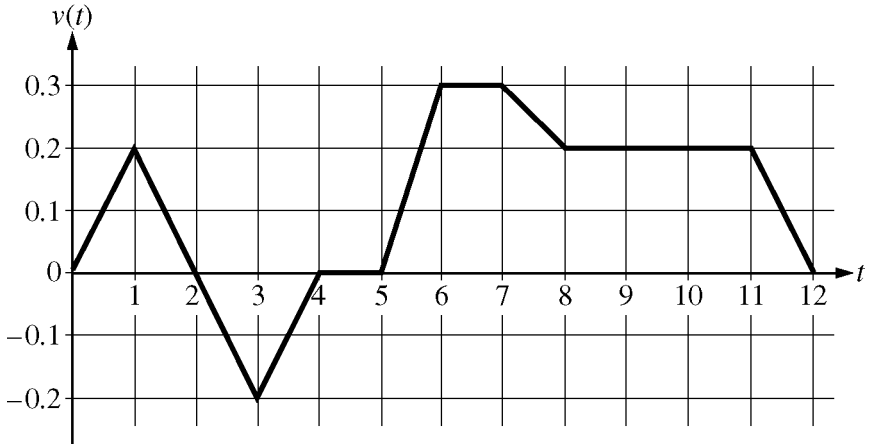
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

   (a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

   (c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

   (d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

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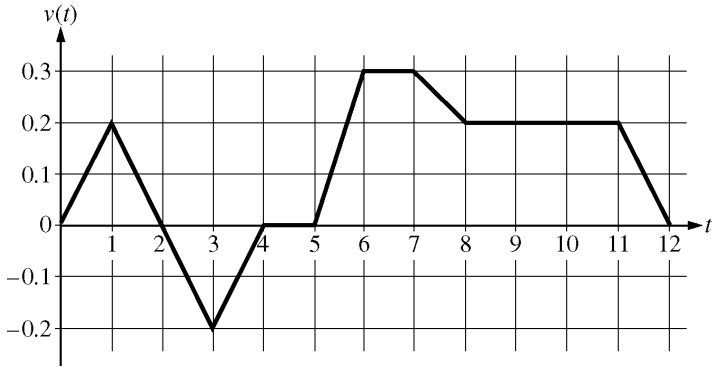
AP® CALCULUS AB
2009 SCORING GUIDELINES

Question 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute^2

2 : { 1 : answer; 1 : units }

(b) $\int_0^{12} |v(t)|\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$.

$$\int_0^{12} |v(t)|\, dt = \int_0^{2} v(t)\, dt - \int_2^{4} v(t)\, dt + \int_4^{12} v(t)\, dt$$
$$= 0.2 + 0.2 + 1.4 = 1.8 \text{ miles}$$

2 : { 1 : meaning of integral; 1 : value of integral }

(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.

2 : { 1 : answer; 1 : reason }

(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school.

$\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school.

Therefore, Caren lives closer to school.

3 : { 2 : Larry's distance from school (1 : integral; 1 : value); 1 : Caren's distance from school and conclusion }

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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1. The rate at which raw sewage enters a treatment tank is given by $E(t) = 850 + 715\cos\left(\frac{\pi t^2}{9}\right)$ gallons per hour for $0 \le t \le 4$ hours. Treated sewage is removed from the tank at the constant rate of 645 gallons per hour. The treatment tank is empty at time $t = 0$.

   (a) How many gallons of sewage enter the treatment tank during the time interval $0 \le t \le 4$? Round your answer to the nearest gallon.

   (b) For $0 \le t \le 4$, at what time t is the amount of sewage in the treatment tank greatest? To the nearest gallon, what is the maximum amount of sewage in the tank? Justify your answers.

   (c) For $0 \le t \le 4$, the cost of treating the raw sewage that enters the tank at time t is $(0.15 - 0.02t)$ dollars per gallon. To the nearest dollar, what is the total cost of treating all the sewage that enters the tank during the time interval $0 \le t \le 4$?

GO ON TO THE NEXT PAGE.

AP® Calculus AB
Free-Response Scoring Guidelines

Question 1

The rate at which raw sewage enters a treatment tank is given by $E(t) = 850 + 715\cos\left(\frac{\pi t^2}{9}\right)$ gallons per hour for $0 \le t \le 4$ hours. Treated sewage is removed from the tank at the constant rate of 645 gallons per hour. The treatment tank is empty at time $t = 0$.

(a) How many gallons of sewage enter the treatment tank during the time interval $0 \le t \le 4$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 4$, at what time t is the amount of sewage in the treatment tank greatest? To the nearest gallon, what is the maximum amount of sewage in the tank? Justify your answers.

(c) For $0 \le t \le 4$, the cost of treating the raw sewage that enters the tank at time t is $(0.15 - 0.02t)$ dollars per gallon. To the nearest dollar, what is the total cost of treating all the sewage that enters the tank during the time interval $0 \le t \le 4$?

(a) $\int_0^4 E(t)\,dt \approx 3981$ gallons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Let $S(t)$ be the amount of sewage in the treatment tank at time t. Then $S'(t) = E(t) - 645$ and $S'(t) = 0$ when $E(t) = 645$. On the interval $0 \le t \le 4$, $E(t) = 645$ when $t = 2.309$ and $t = 3.559$.

t (hours)	amount of sewage in treatment tank
0	0
2.309	$\int_0^{2.309} E(t)\,dt - 645(2.309) = 1637.178$
3.559	$\int_0^{3.559} E(t)\,dt - 645(3.559) = 1228.520$
4	$3981.022 - 645(4) = 1401.022$

The amount of sewage in the treatment tank is greatest at $t = 2.309$ hours. At that time, the amount of sewage in the tank, rounded to the nearest gallon, is 1637 gallons.

$4 : \begin{cases} 1 : \text{sets } E(t) = 645 \\ 1 : \text{identifies } t = 2.309 \text{ as a candidate} \\ 1 : \text{amount of sewage at } t = 2.309 \\ 1 : \text{conclusion} \end{cases}$

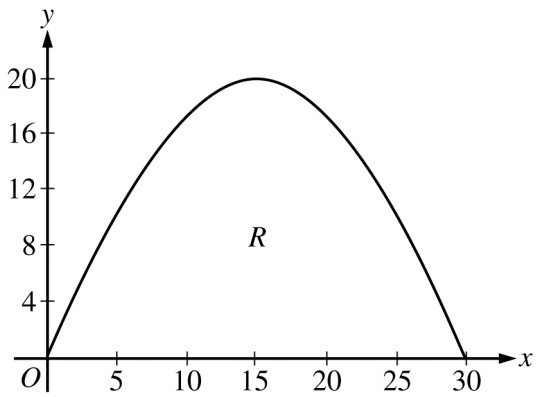
(c) Total cost $= \int_0^4 (0.15 - 0.02t)E(t)\,dt = 474.320$

The total cost of treating the sewage that enters the tank during the time interval $0 \le t \le 4$, to the nearest dollar, is \$474.

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.

   (a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

   (b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

   (c) Find the perimeter of the base of the cake.

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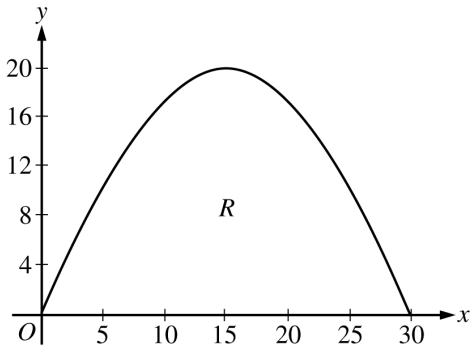
AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

Question 1

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.



(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

(c) Find the perimeter of the base of the cake.

(a) Area $= 30 \cdot 20 - \int_0^{30} f(x)\,dx = 218.028$ cm^2

3 : { 2 : integral; 1 : answer }

(b) Volume $= \int_0^{30} \frac{\pi}{2}\left(\frac{f(x)}{2}\right)^2 dx = 2356.194$ cm^3

Therefore, the baker needs $2356.194 \times 0.05 = 117.809$ or 117.810 grams of chocolate.

3 : { 2 : integral; 1 : answer }

(c) Perimeter $= 30 + \int_0^{30} \sqrt{1 + (f'(x))^2}\,dx = 81.803$ or 81.804 cm

3 : { 2 : integral; 1 : answer }

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2. The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

   (a) How many people are in the auditorium when the concert begins?

   (b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

   (c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

   (d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

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2009 SCORING GUIDELINES

Question 2

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

(a) $\int_0^2 R(t)\,dt = 980$ people

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$
The maximum rate may occur at 0, $a = 1.36296$, or 2.

$R(0) = 0$
$R(a) = 854.527$
$R(2) = 120$

The maximum rate occurs when $t = 1.362$ or 1.363.

$3 : \begin{cases} 1 : \text{considers } R'(t) = 0 \\ 1 : \text{interior critical point} \\ 1 : \text{answer and justification} \end{cases}$

(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2 - t)R(t)\,dt = 387.5$
The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $\frac{1}{980}w(2) = \frac{1}{980}\int_0^2 (2 - t)R(t)\,dt = 0.77551$
On average, a person waits 0.775 or 0.776 hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t > 0$, where

$$\frac{dx}{dt} = \left(\frac{6}{t} - 3\right)^{1/3} \text{ and } \frac{dy}{dt} = te^{-t}.$$

At time $t = 3$, the particle is at the point $(5, 4)$.

(a) Find the speed of the particle at time $t = 3$.

(b) Write an equation for the line tangent to the path of the particle at time $t = 3$.

(c) Is there a time t at which the particle is farthest to the right? If yes, explain why and give the value of t and the x-coordinate of the position of the particle at that time. If no, explain why not.

(d) Describe the behavior of the path of the particle as t increases without bound.

GO ON TO THE NEXT PAGE.

AP® Calculus BC
Free-Response Scoring Guidelines

Question 2

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t > 0$, where

$$\frac{dx}{dt} = \left(\frac{6}{t} - 3\right)^{1/3} \text{ and } \frac{dy}{dt} = te^{-t}.$$

At time $t = 3$, the particle is at the point $(5, 4)$.

(a) Find the speed of the particle at time $t = 3$.

(b) Write an equation for the line tangent to the path of the particle at time $t = 3$.

(c) Is there a time t at which the particle is farthest to the right? If yes, explain why and give the value of t and the x-coordinate of the position of the particle at that time. If no, explain why not.

(d) Describe the behavior of the path of the particle as t increases without bound.

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 1.011$

1 : answer

(b) At $t = 3$, $\dfrac{dy}{dx} = \dfrac{y'(3)}{x'(3)} = \dfrac{3e^{-3}}{-1} = -\dfrac{3}{e^3}$
$= -0.149$

An equation for the line tangent to the path at the point $(5, 4)$ is $y - 4 = -\dfrac{3}{e^3}(x - 5)$.

2 : { 1 : slope; 1 : equation of tangent line }

(c) Since $\dfrac{dx}{dt} > 0$ for $0 < t < 2$ and $\dfrac{dx}{dt} < 0$ for $t > 2$, the particle is farthest to the right at $t = 2$. The x-coordinate of the position is

$$x(2) = 5 + \int_3^2 \left(\frac{6}{t} - 3\right)^{1/3} dt = 5.791 \text{ or } 5.792$$

4 : { 1 : "yes" with explanation; 1 : $t = 2$; 2 : { 1 : integral for $x(2)$; 1 : x-coordinate } }

(d) Since $\lim\limits_{t\to\infty} \dfrac{dx}{dt} = (-3)^{1/3}$ and $\lim\limits_{t\to\infty} \dfrac{dy}{dt} = 0$, the particle will continue moving farther to the left while approaching a horizontal asymptote.

2 : { 1 : behavior of x; 1 : behavior of y }

2. A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

   (a) What was the distance between the road and the edge of the water at the end of the storm?

   (b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

   (c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

   (d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

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2009 SCORING GUIDELINES (Form B)

Question 2

A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

(a) What was the distance between the road and the edge of the water at the end of the storm?

(b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of 1.007 meters/hours^2.

$2 : \begin{cases} 1 : \text{interpretation of } f'(4) \\ 1 : \text{units} \end{cases}$

(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$
The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5.

$f(0) = -2$
$f(0.66187) = -1.39760$
$f(2.84038) = -2.26963$
$f(5) = -0.48027$

The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.

$3 : \begin{cases} 1 : \text{considers } f'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$

$2 : \begin{cases} 1 : \text{integral of } g \\ 1 : \text{answer} \end{cases}$

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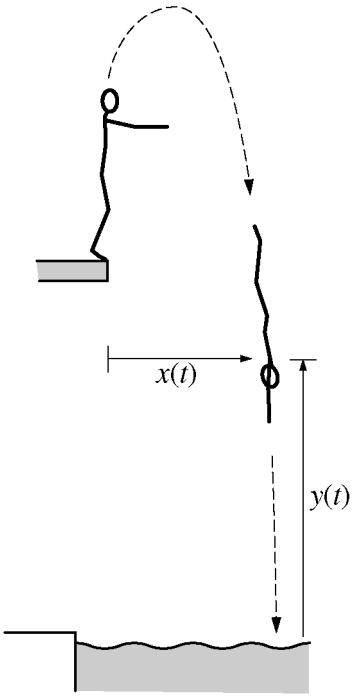

Note: Figure not drawn to scale.

3. A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

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END OF PART A OF SECTION II

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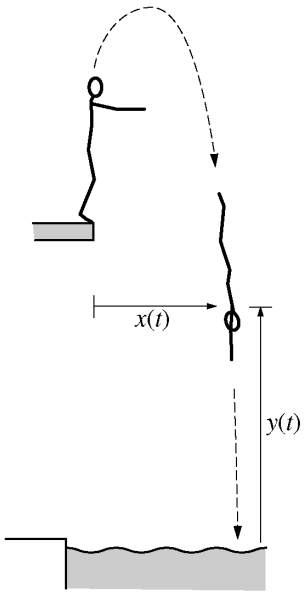
Question 3

A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.



Note: Figure not drawn to scale.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

(a) $\frac{dy}{dt} = 0$ only when $t = 0.36735$. Let $b = 0.36735$.
The maximum vertical distance from the water surface to the diver's shoulders is
$y(b) = 11.4 + \int_0^b \frac{dy}{dt}\,dt = 12.061$ meters.

Alternatively, $y(t) = 11.4 + 3.6t - 4.9t^2$, so $y(b) = 12.061$ meters.

3 : { 1 : considers $\frac{dy}{dt} = 0$; 1 : integral or $y(t)$; 1 : answer }

(b) $y(A) = 11.4 + \int_0^A \frac{dy}{dt}\,dt = 11.4 + 3.6A - 4.9A^2 = 0$ when $A = 1.936$ seconds.

2 : { 1 : equation; 1 : answer }

(c) $\int_0^A \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 12.946$ meters

2 : { 1 : integral; 1 : answer }

(d) At time A, $\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=A} = -19.21913$.

The angle between the path of the diver and the water is $\tan^{-1}(19.21913) = 1.518$ or 1.519.

2 : { 1 : $\frac{dy}{dx}$ at time A; 1 : answer }

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

---

t (minutes)	0	4	8	12	16
$H(t)$ (°C)	65	68	73	80	90

3. The temperature, in degrees Celsius (°C), of an oven being heated is modeled by an increasing differentiable function H of time t, where t is measured in minutes. The table above gives the temperature as recorded every 4 minutes over a 16-minute period.

   (a) Use the data in the table to estimate the instantaneous rate at which the temperature of the oven is changing at time $t = 10$. Show the computations that lead to your answer. Indicate units of measure.

   (b) Write an integral expression in terms of H for the average temperature of the oven between time $t = 0$ and time $t = 16$. Estimate the average temperature of the oven using a left Riemann sum with four subintervals of equal length. Show the computations that lead to your answer.

   (c) Is your approximation in part (b) an underestimate or an overestimate of the average temperature? Give a reason for your answer.

   (d) Are the data in the table consistent with or do they contradict the claim that the temperature of the oven is increasing at an increasing rate? Give a reason for your answer.

AP® Calculus AB
Free-Response Scoring Guidelines

Question 3

t (minutes)	0	4	8	12	16
$H(t)$ (°C)	65	68	73	80	90

The temperature, in degrees Celsius $(°C)$, of an oven being heated is modeled by an increasing differentiable function H of time t, where t is measured in minutes. The table above gives the temperature as recorded every 4 minutes over a 16-minute period.

(a) Use the data in the table to estimate the instantaneous rate at which the temperature of the oven is changing at time $t = 10$. Show the computations that lead to your answer. Indicate units of measure.

(b) Write an integral expression in terms of H for the average temperature of the oven between time $t = 0$ and time $t = 16$. Estimate the average temperature of the oven using a left Riemann sum with four subintervals of equal length. Show the computations that lead to your answer.

(c) Is your approximation in part (b) an underestimate or an overestimate of the average temperature? Give a reason for your answer.

(d) Are the data in the table consistent with or do they contradict the claim that the temperature of the oven is increasing at an increasing rate? Give a reason for your answer.

(a) $H'(10) \approx \dfrac{H(12) - H(8)}{12 - 8} = \dfrac{80 - 73}{4} = \dfrac{7}{4}°\text{C/min}$

2 : { 1 : difference quotient; 1 : answer with units }

(b) Average temperature is $\dfrac{1}{16}\displaystyle\int_0^{16} H(t)\,dt$

$\displaystyle\int_0^{16} H(t)\,dt \approx 4 \cdot (65 + 68 + 73 + 80)$

Average temperature $\approx \dfrac{4 \cdot 286}{16} = 71.5°\text{C}$

3 : { 1 : $\dfrac{1}{16}\displaystyle\int_0^{16} H(t)\,dt$; 1 : left Riemann sum; 1 : answer }

(c) The left Riemann sum approximation is an underestimate of the integral because the graph of H is increasing. Dividing by 16 will not change the inequality, so $71.5°\text{C}$ is an underestimate of the average temperature.

1 : answer with reason

(d) If a continuous function is increasing at an increasing rate, then the slopes of the secant lines of the graph of the function are increasing. The slopes of the secant lines for the four intervals in the table are $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, and $\frac{10}{4}$, respectively.

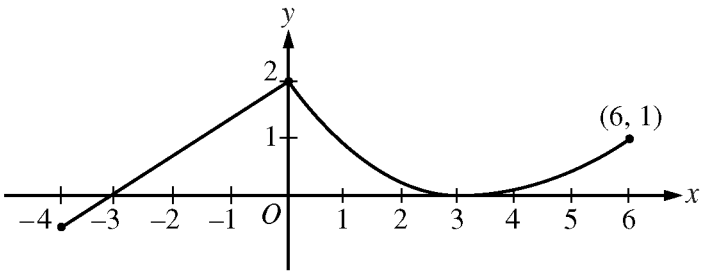
Since the slopes are increasing, the data are consistent with the claim.

OR

By the Mean Value Theorem, the slopes are also the values of $H'(c_k)$ for some times $c_1 < c_2 < c_3 < c_4$, respectively.

Since these derivative values are positive and increasing, the data are consistent with the claim.

3 : { 1 : considers slopes of four secant lines; 1 : explanation; 1 : conclusion consistent with explanation }



Graph of f

3. A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

   (a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

   (b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

   (c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

   (d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

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END OF PART A OF SECTION II

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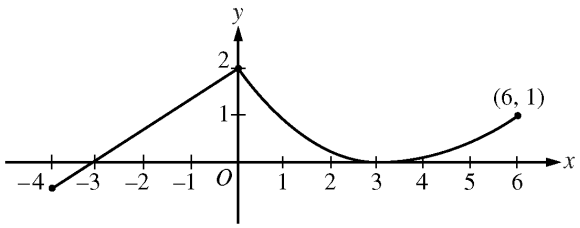
Question 3



Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

(a) $\lim\limits_{h \to 0^-} \dfrac{f(h) - f(0)}{h} = \dfrac{2}{3}$

$\lim\limits_{h \to 0^+} \dfrac{f(h) - f(0)}{h} < 0$

Since the one-sided limits do not agree, f is not differentiable at $x = 0$.

$2 : \begin{cases} 1 : \text{sets up difference quotient at } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $\dfrac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.

$2 : \begin{cases} 1 : \text{expression for average rate of change} \\ 1 : \text{answer with reason} \end{cases}$

(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$.

Also, $\dfrac{f(6) - f(3)}{6 - 3} = \dfrac{1 - 0}{6 - 3} = \dfrac{1}{3}$.

By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \dfrac{1}{3}$.

$2 : \begin{cases} 1 : \text{answers "yes" and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$

(d) $g'(x) = f(x)$, $g''(x) = f'(x)$

$g''(x) > 0$ when $f'(x) > 0$

This is true for $-4 < x < 0$ and $3 < x < 6$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{considers } g''(x) > 0 \\ 1 : \text{answer} \end{cases}$

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

4. Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

   (a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

   (b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

   (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

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Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a) $f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$

$= 2 + 4 \cdot \frac{1}{2} = 4$

$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$

$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c) $\frac{dy}{dx} = x^2(6 - y)$

$\int \frac{1}{6-y}\,dy = \int x^2\,dx$

$-\ln|6 - y| = \frac{1}{3}x^3 + C$

$-\ln 4 = -\frac{1}{3} + C$

$C = \frac{1}{3} - \ln 4$

$\ln|6 - y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$

$|6 - y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$

$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

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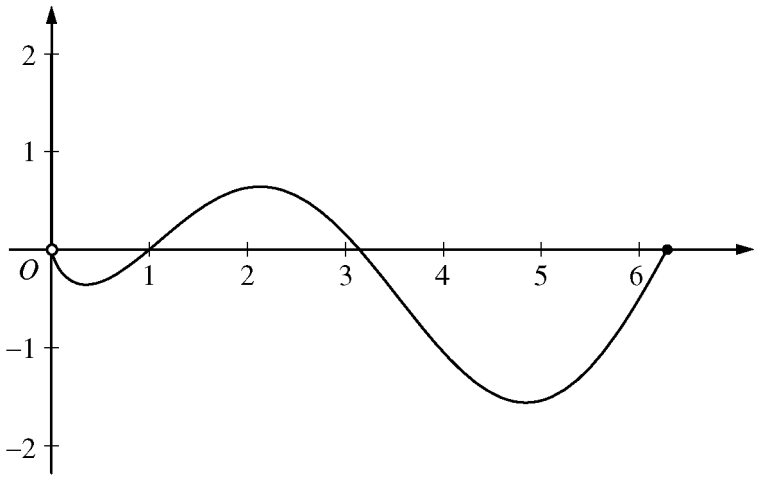


Graph of f

4. Let f be the function given by $f(x) = (\ln x)(\sin x)$. The figure above shows the graph of f for $0 < x \le 2\pi$.

   The function g is defined by $g(x) = \int_1^x f(t)\,dt$ for $0 < x \le 2\pi$.

   (a) Find $g(1)$ and $g'(1)$.

   (b) On what intervals, if any, is g increasing? Justify your answer.

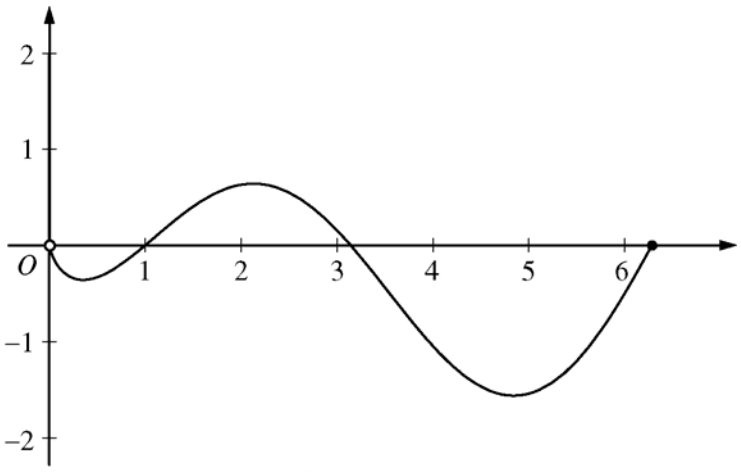
   (c) For $0 < x \le 2\pi$, find the value of x at which g has an absolute minimum. Justify your answer.

   (d) For $0 < x < 2\pi$, is there a value of x at which the graph of g is tangent to the x-axis? Explain why or why not.

GO ON TO THE NEXT PAGE.

AP® Calculus AB
Free-Response Scoring Guidelines

Question 4



Graph of f

Let f be the function given by $f(x) = (\ln x)(\sin x)$. The figure above shows the graph of f for $0 < x \le 2\pi$. The function g is defined by $g(x) = \int_1^x f(t)\,dt$ for $0 < x \le 2\pi$.

(a) Find $g(1)$ and $g'(1)$.

(b) On what intervals, if any, is g increasing? Justify your answer.

(c) For $0 < x \le 2\pi$, find the value of x at which g has an absolute minimum. Justify your answer.

(d) For $0 < x < 2\pi$, is there a value of x at which the graph of g is tangent to the x-axis? Explain why or why not.

(a) $g(1) = \int_1^1 f(t)\,dt = 0$ and $g'(1) = f(1) = 0$

$2 : \begin{cases} 1 : g(1) \\ 1 : g'(1) \end{cases}$

(b) Since $g'(x) = f(x)$, g is increasing on the interval $1 \le x \le \pi$ because $f(x) > 0$ for $1 < x < \pi$.

$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

(c) For $0 < x < 2\pi$, $g'(x) = f(x) = 0$ when $x = 1,\ \pi$. $g' = f$ changes from negative to positive only at $x = 1$. The absolute minimum must occur at $x = 1$ or at the right endpoint. Since $g(1) = 0$ and $g(2\pi) = \int_1^{2\pi} f(t)\,dt = \int_1^{\pi} f(t)\,dt + \int_{\pi}^{2\pi} f(t)\,dt < 0$ by comparison of the two areas, the absolute minimum occurs at $x = 2\pi$.

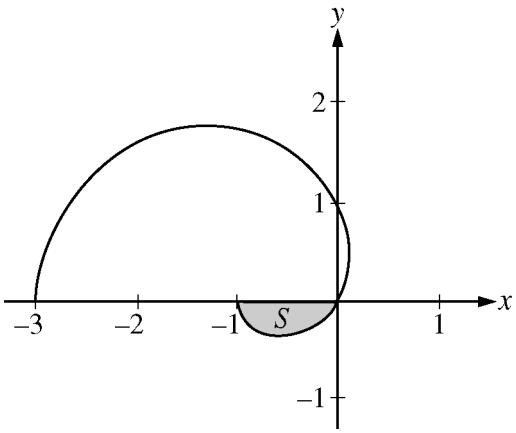
$3 : \begin{cases} 1 : \text{identifies 1 and } 2\pi \text{ as candidates} \\ \quad\quad \text{- or -} \\ \quad \text{indicates that the graph of } g \\ \quad \text{decreases, increases, then decreases} \\ 1 : \text{justifies } g(2\pi) < g(1) \\ 1 : \text{answer} \end{cases}$

(d) Yes, the graph of g is tangent to the x-axis at $x = 1$ since $g(1) = 0$ and $g'(1) = 0$.

$2 : \begin{cases} 1 : \text{answer of ``yes'' with } x = 1 \\ 1 : \text{explanation} \end{cases}$

CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



4. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

   (a) Write an integral expression for the area of S.

   (b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

   (c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

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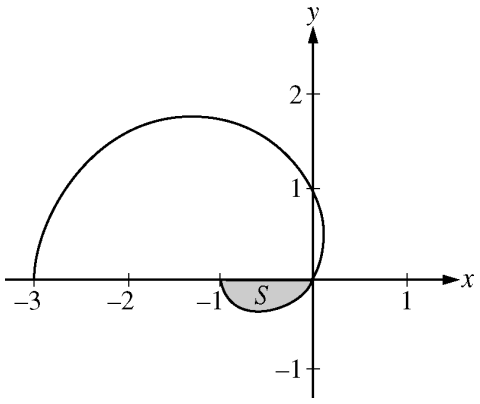
Question 4

The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.



(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$\frac{dr}{d\theta} = 2\sin\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$

$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0$, $y = 1$.

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$

The tangent line is given by $y = 1 - 2x$.

$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

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x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

5. Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

   (a) Estimate $f'(4)$. Show the work that leads to your answer.

   (b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

   (c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

   (d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

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AP® CALCULUS AB
2009 SCORING GUIDELINES

Question 5

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

1 : answer

(b) $\displaystyle\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$

$= 3(13 - 2) - 5(f(13) - f(2)) = 8$

2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }

(c) $\displaystyle\int_2^{13} f(x)\,dx \approx f(2)(3 - 2) + f(3)(5 - 3) + f(5)(8 - 5) + f(8)(13 - 8) = 18$

2 : { 1 : left Riemann sum; 1 : answer }

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$.
Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$.
Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$.
Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.
Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.

4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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5. Let f be the function satisfying $f'(x) = 4x - 2xf(x)$ for all real numbers x, with $f(0) = 5$ and $\lim_{x \to \infty} f(x) = 2$.

   (a) Find the value of $\int_0^{\infty} (4x - 2xf(x))\, dx$. Show the work that leads to your answer.

   (b) Use Euler's method to approximate $f(-1)$, starting at $x = 0$, with two steps of equal size.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 4x - 2xy$ with the initial condition $f(0) = 5$.

GO ON TO THE NEXT PAGE.

AP® Calculus BC
Free-Response Scoring Guidelines

Question 5

Let f be the function satisfying $f'(x) = 4x - 2xf(x)$ for all real numbers x, with $f(0) = 5$ and $\lim_{x\to\infty} f(x) = 2$.

(a) Find the value of $\int_0^\infty (4x - 2xf(x))\,dx$. Show the work that leads to your answer.

(b) Use Euler's method to approximate $f(-1)$, starting at $x = 0$, with two steps of equal size.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 4x - 2xy$ with the initial condition $f(0) = 5$.

(a) $\int_0^\infty (4x - 2xf(x))\,dx$

$= \int_0^\infty f'(x)\,dx = \lim_{b\to\infty} \int_0^b f'(x)\,dx = \lim_{b\to\infty} \left[f(x)\Big|_0^b \right]$

$= \lim_{b\to\infty} (f(b) - f(0)) = 2 - 5 = -3$

2 : { 1 : use of FTC; 1 : answer from limiting process }

(b) $f\left(-\frac{1}{2}\right) \approx f(0) + f'(0)\cdot\left(-\frac{1}{2}\right) = 5 + 0\cdot\left(-\frac{1}{2}\right) = 5$

$f(-1) \approx f\left(-\frac{1}{2}\right) + f'\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right) = 5 + 3\cdot\left(-\frac{1}{2}\right) = \frac{7}{2}$

2 : { 1 : Euler's method with two steps; 1 : Euler's approximation to $f(-1)$ }

(c) $\frac{1}{4-2y}\,dy = x\,dx$

$-\frac{1}{2}\ln|4-2y| = \frac{1}{2}x^2 + A$

$\ln|4-2y| = -x^2 + B$

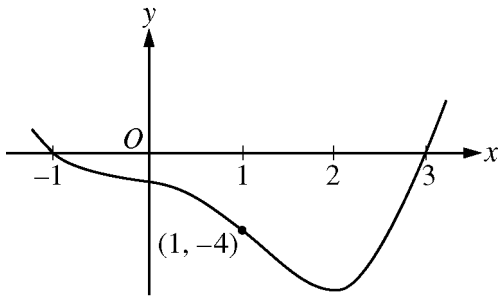
$2y - 4 = Ce^{-x^2}$

$C = 6$

$y = 2 + 3e^{-x^2}$ for all real numbers x

5 : { 1 : separates variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for y }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration



Graph of f'

5. Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.

   (a) Write an equation for the line tangent to the graph of g at $x = 1$.

   (b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

   (c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

   (d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

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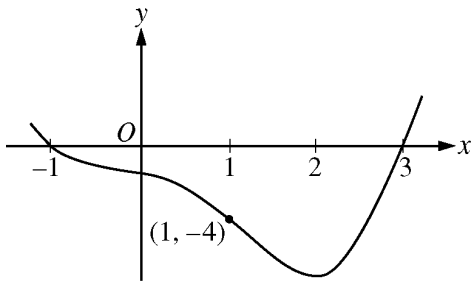
AP® CALCULUS AB
2009 SCORING GUIDELINES (Form B)

Question 5

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.



Graph of f'

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)}f'(x)$, $g'(1) = e^{f(1)}f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x - 1)$.

$3 : \begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)}f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $\dfrac{g'(3) - g'(1)}{3 - 1} = \dfrac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

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6. The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

   (b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

   (d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

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END OF EXAM

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Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(c) $\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$

Therefore, the interval of convergence is $(-\infty, \infty)$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{answer} \end{cases}$

(d) $f''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$

Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.

$2 : \begin{cases} 1 : f''(x) \\ 1 : \text{answer} \end{cases}$

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CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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6. The function g is continuous for all real numbers x and is defined by

$$g(x) = \frac{\cos(2x) - 1}{x^2} \text{ for } x \neq 0.$$

(a) Use L'Hospital's Rule to find the value of $g(0)$. Show the work that leads to your answer.

(b) Let f be the function given by $f(x) = \cos(2x)$. Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(c) Use your answer from part (b) to write the first three nonzero terms and the general term of the Taylor series for g about $x = 0$.

(d) Determine whether g has a relative minimum, a relative maximum, or neither at $x = 0$. Justify your answer.

GO ON TO THE NEXT PAGE.

AP® Calculus BC
Free-Response Scoring Guidelines

Question 6

The function g is continuous for all real numbers x and is defined by

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(a) Use L'Hospital's Rule to find the value of $g(0)$. Show the work that leads to your answer.

(b) Let f be the function given by $f(x) = \cos(2x)$. Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(c) Use your answer from part (b) to write the first three nonzero terms and the general term of the Taylor series for g about $x = 0$.

(d) Determine whether g has a relative minimum, a relative maximum, or neither at $x = 0$. Justify your answer.

(a) Since g is continuous,

$$g(0) = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\cos(2x) - 1}{x^2}$$
$$= \lim_{x\to 0} \frac{-2\sin(2x)}{2x}$$
$$= \lim_{x\to 0} \frac{-4\cos(2x)}{2} = -2$$

2 : { 1 : uses L'Hospital's Rule correctly at least once; 1 : answer }

(b) $$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$
$$= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \frac{64}{6!}x^6 + \cdots + (-1)^n \frac{2^{2n}}{(2n)!}x^{2n} + \cdots$$

3 : { 1 : first two terms; 1 : next two terms; 1 : general term }

(c) $$g(x) = -\frac{4}{2!} + \frac{16}{4!}x^2 - \frac{64}{6!}x^4 + \cdots + (-1)^n \frac{2^{2n}}{(2n)!}x^{2n-2} + \cdots$$

2 : { 1 : first three terms; 1 : general term }

(d) $g'(x) = \frac{2 \cdot 16}{4!}x - \frac{4 \cdot 64}{6!}x^3 + \cdots$, so $g'(0) = 0$.

$g''(x) = \frac{2 \cdot 16}{4!} - \frac{3 \cdot 4 \cdot 64}{6!}x^2 + \cdots$, so $g''(0) > 0$.

Therefore, g has a relative minimum at $x = 0$ by the Second Derivative Test.

2 : { 1 : answer; 1 : justification }

6. The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

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END OF EXAM

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AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$.
The series converges if and only if $|x+1| < 1$.
Therefore, the interval of convergence is $-2 < x < 0$.

3 : { 1 : identifies as geometric; 1 : $|x+1| < 1$; 1 : interval of convergence }

OR

OR

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

3 : { 1 : sets up limit of ratio; 1 : radius of convergence; 1 : interval of convergence }

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

(b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

1 : answer

(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

2 : { 1 : antiderivative; 1 : value }

(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$

3 : { 1 : first three terms; 1 : general term; 1 : value of $h\left(\frac{1}{2}\right)$ }

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CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

1. There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9 \end{cases}.$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

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Question 1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

(a) $\int_0^6 f(t)\,dt = 142.274$ or 142.275 cubic feet

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Rate of change is $f(8) - g(8) = -59.582$ or -59.583 cubic feet per hour.

1 : answer

(c) $h(0) = 0$

For $0 < t \le 6$, $h(t) = h(0) + \int_0^t g(s)\,ds = 0 + \int_0^t 0\,ds = 0$.

For $6 < t \le 7$, $h(t) = h(6) + \int_6^t g(s)\,ds = 0 + \int_6^t 125\,ds = 125(t-6)$.

For $7 < t \le 9$, $h(t) = h(7) + \int_7^t g(s)\,ds = 125 + \int_7^t 108\,ds = 125 + 108(t-7)$.

Thus, $h(t) = \begin{cases} 0 & \text{for } 0 \le t \le 6 \\ 125(t-6) & \text{for } 6 < t \le 7 \\ 125 + 108(t-7) & \text{for } 7 < t \le 9 \end{cases}$

3 : $\begin{cases} 1 : h(t) \text{ for } 0 \le t \le 6 \\ 1 : h(t) \text{ for } 6 < t \le 7 \\ 1 : h(t) \text{ for } 7 < t \le 9 \end{cases}$

(d) Amount of snow is $\int_0^9 f(t)\,dt - h(9) = 26.334$ or 26.335 cubic feet.

3 : $\begin{cases} 1 : \text{integral} \\ 1 : h(9) \\ 1 : \text{answer} \end{cases}$

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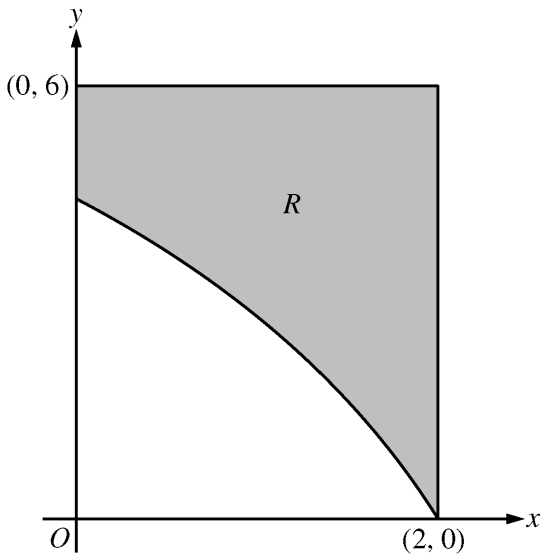
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

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2010 SCORING GUIDELINES (Form B)

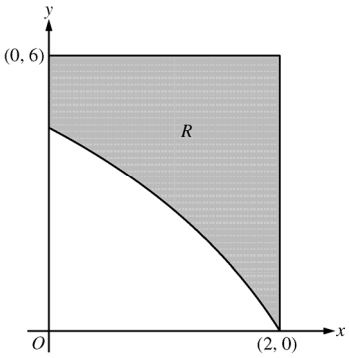
Question 1

In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.



	1 : Correct limits in an integral in (a), (b), or (c)
(a) $\int_0^2 (6 - 4\ln(3 - x))\,dx = 6.816$ or 6.817	2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $\pi\int_0^2 \left((8 - 4\ln(3 - x))^2 - (8 - 6)^2\right) dx$ $= 168.179$ or 168.180	3 : $\begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) $\int_0^2 (6 - 4\ln(3 - x))^2\,dx = 26.266$ or 26.267	3 : $\begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

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t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

2. A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

   (a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

   (b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$.

   Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

   (c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

   (d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

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2010 SCORING GUIDELINES

Question 2

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$.

Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$\frac{1}{8}\left(2 \cdot \dfrac{E(0) + E(2)}{2} + 3 \cdot \dfrac{E(2) + E(5)}{2} + 2 \cdot \dfrac{E(5) + E(7)}{2} + 1 \cdot \dfrac{E(7) + E(8)}{2}\right)$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

3 : { 1 : trapezoidal sum; 1 : approximation; 1 : meaning }

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

2 : { 1 : integral; 1 : answer }

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3 : { 1 : considers $P'(t) = 0$; 1 : identifies candidates; 1 : answer with justification }

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2. The velocity vector of a particle moving in the xy-plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

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2010 SCORING GUIDELINES (Form B)

Question 2

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

(a) The tangent line is vertical when $x'(t) = 0$ and $y'(t) \neq 0$.

On $0 < t < 1.5$, this happens at $t = 1.253$ and $t = 1.144$ or 1.145.

$2 : \begin{cases} 1 : \text{sets } \frac{dx}{dy} = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} = 0.863447$

$x(1) = -2 + \int_0^1 x'(t)\,dt = 9.314695$

$y(1) = 3 + \int_0^1 y'(t)\,dt = 4.620537$

The line tangent to the path of the particle at $t = 1$ has equation $y = 4.621 + 0.863(x - 9.315)$.

$4 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{t=1} \\ 1 : x(1) \\ 1 : y(1) \\ 1 : \text{equation} \end{cases}$

(c) Speed $= \sqrt{(x'(1))^2 + (y'(1))^2} = 4.105$

1 : answer

(d) Acceleration vector: $\langle x''(1), y''(1) \rangle = \langle -28.425, 2.161 \rangle$

$2 : \begin{cases} 1 : x''(1) \\ 1 : y''(1) \end{cases}$

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3. A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

   (a) Find the speed of the particle at time $t = 3$ seconds.

   (b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

   (c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

   (d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

      (i) The two values of t when that occurs

      (ii) The slopes of the lines tangent to the particle's path at that point

      (iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

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END OF PART A OF SECTION II

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Question 3

A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.
(i) The two values of t when that occurs
(ii) The slopes of the lines tangent to the particle's path at that point
(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 2.828$ meters per second

1 : answer

(b) $x'(t) = 2t - 4$

Distance $= \int_0^4 \sqrt{(2t-4)^2 + \left(te^{t-3} - 1\right)^2}\, dt = 11.587$ or 11.588 meters

2 : { 1 : integral; 1 : answer }

(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ when $te^{t-3} - 1 = 0$ and $2t - 4 \ne 0$

This occurs at $t = 2.20794$.

Since $x'(2.20794) > 0$, the particle is moving toward the right at time $t = 2.207$ or 2.208.

3 : { 1 : considers $\frac{dy}{dx} = 0$; 1 : $t = 2.207$ or 2.208; 1 : direction of motion with reason }

(d) $x(t) = 5$ at $t = 1$ and $t = 3$

At time $t = 1$, the slope is $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = 0.432$.

At time $t = 3$, the slope is $\left.\frac{dy}{dx}\right|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right|_{t=3} = 1$.

$y(1) = y(3) = 3 + \frac{1}{e} + \int_2^3 \frac{dy}{dt}\, dt = 4$

3 : { 1 : $t = 1$ and $t = 3$; 1 : slopes; 1 : y-coordinate }

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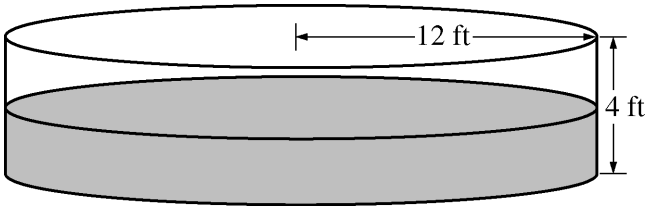
t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



3. The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

   (a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

   (b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

   (c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

   (d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

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END OF PART A OF SECTION II

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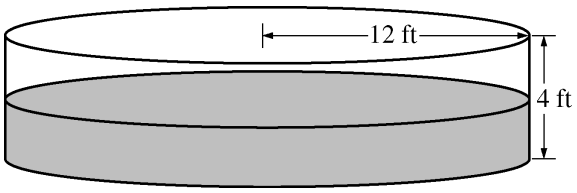
2010 SCORING GUIDELINES (Form B)

Question 3

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

(b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

(c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

(d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

(a) $\displaystyle\int_0^{12} P(t)\,dt \approx 46 \cdot 4 + 57 \cdot 4 + 62 \cdot 4 = 660 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{answer} \end{cases}$

(b) $\displaystyle\int_0^{12} R(t)\,dt = 225.594 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $1000 + \displaystyle\int_0^{12} P(t)\,dt - \int_0^{12} R(t)\,dt = 1434.406$

At time $t = 12$ hours, the volume of water in the pool is approximately 1434 ft^3.

1 : answer

(d) $V'(t) = P(t) - R(t)$

$V'(8) = P(8) - R(8) = 60 - 25e^{-0.4} = 43.241$ or $43.242 \text{ ft}^3/\text{hr}$

$V = \pi(12)^2 h$

$\dfrac{dV}{dt} = 144\pi \dfrac{dh}{dt}$

$\left.\dfrac{dh}{dt}\right|_{t=8} = \dfrac{1}{144\pi} \cdot \left.\dfrac{dV}{dt}\right|_{t=8} = 0.095$ or $0.096 \text{ ft}/\text{hr}$

$4 : \begin{cases} 1 : V'(8) \\ 1 : \text{equation relating } \dfrac{dV}{dt} \text{ and } \dfrac{dh}{dt} \\ 1 : \left.\dfrac{dh}{dt}\right|_{t=8} \\ 1 : \text{units of } \text{ft}^3/\text{hr} \text{ and } \text{ft}/\text{hr} \end{cases}$

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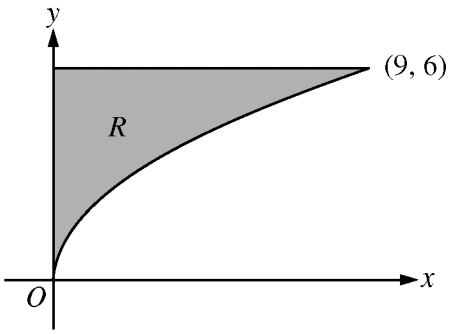
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



4. Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

   (c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

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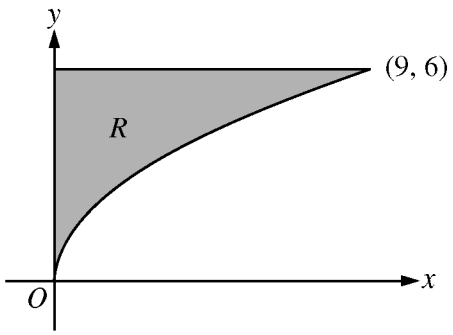
Question 4



Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(a) Area $= \int_0^9 \left(6 - 2\sqrt{x}\right) dx = \left(6x - \frac{4}{3}x^{3/2}\right)\Big|_{x=0}^{x=9} = 18$

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(b) Volume $= \pi\int_0^9 \left(\left(7 - 2\sqrt{x}\right)^2 - (7-6)^2\right) dx$

3 : { 2 : integrand; 1 : limits and constant }

(c) Solving $y = 2\sqrt{x}$ for x yields $x = \frac{y^2}{4}$.

Each rectangular cross section has area $\left(3\frac{y^2}{4}\right)\left(\frac{y^2}{4}\right) = \frac{3}{16}y^4$.

Volume $= \int_0^6 \frac{3}{16}y^4\, dy$

3 : { 2 : integrand; 1 : answer }

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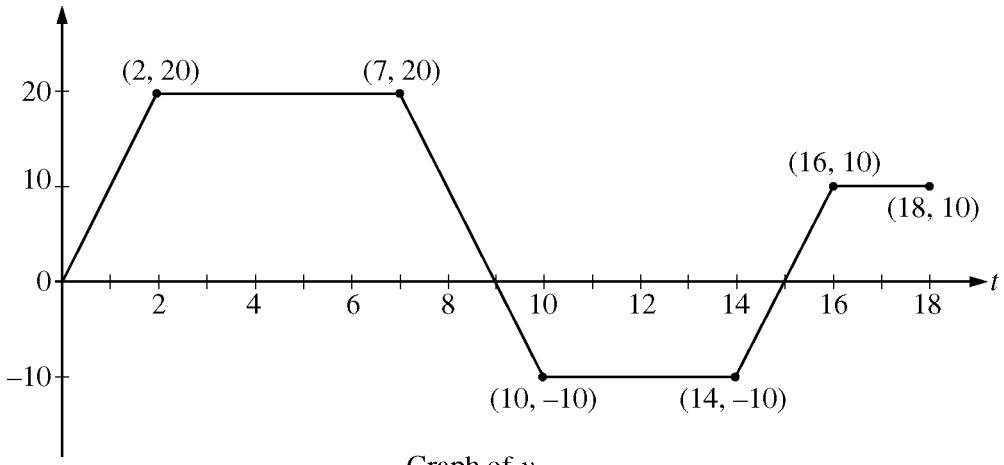
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of v

4. A squirrel starts at building A at time $t = 0$ and travels along a straight, horizontal wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

   (a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

   (b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A ? How far from building A is the squirrel at that time?

   (c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

   (d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

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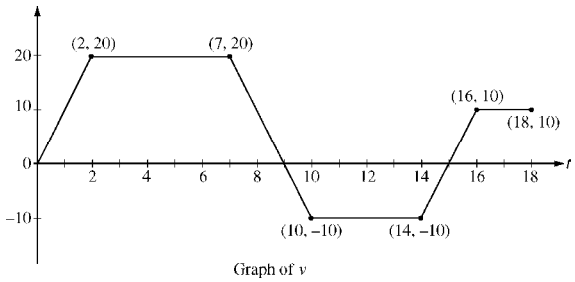
Question 4

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



Graph of v

(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at this time?

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

(a) The squirrel changes direction whenever its velocity changes sign. This occurs at $t = 9$ and $t = 15$.

$2 : \begin{cases} 1 : t\text{-values} \\ 1 : \text{explanation} \end{cases}$

(b) Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

t	position at time t
0	0
9	$\frac{9+5}{2} \cdot 20 = 140$
15	$140 - \frac{6+4}{2} \cdot 10 = 90$
18	$90 + \frac{3+2}{2} \cdot 10 = 115$

The squirrel is farthest from building A at time $t = 9$; its greatest distance from the building is 140.

$2 : \begin{cases} 1 : \text{identifies candidates} \\ 1 : \text{answers} \end{cases}$

(c) The total distance traveled is $\int_0^{18} |v(t)|\, dt = 140 + 50 + 25 = 215$.

1 : answer

(d) For $7 < t < 10$, $a(t) = \frac{20 - (-10)}{7 - 10} = -10$

$v(t) = 20 - 10(t - 7) = -10t + 90$

$x(7) = \frac{7+5}{2} \cdot 20 = 120$

$x(t) = x(7) + \int_7^t (-10u + 90)\, du$

$= 120 + \left(-5u^2 + 90u\right)\Big|_{u=7}^{u=t}$

$= -5t^2 + 90t - 265$

$4 : \begin{cases} 1 : a(t) \\ 1 : v(t) \\ 2 : x(t) \end{cases}$

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5. Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

   (a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

   (b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

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Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a) $f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right|_{(1,\,0)}\right) \cdot \Delta x$

$= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$

$f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\right) \cdot \Delta x$

$\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} \left(x^3 - 1\right)$ and we may apply L'Hospital's Rule.

$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$

$2 : \begin{cases} 1 : \text{use of L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = 1 - y$

$\int \frac{1}{1 - y}\, dy = \int 1\, dx$

$-\ln|1 - y| = x + C$

$-\ln 1 = 1 + C \Rightarrow C = -1$

$\ln|1 - y| = 1 - x$

$|1 - y| = e^{1-x}$

$f(x) = 1 - e^{1-x}$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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5. Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1 + 4x^2}$, for all $x > 0$.

   (a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

   (b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

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Question 5

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1+4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

(a) $g'(x) = \dfrac{4\left(1+4x^2\right) - 4x(8x)}{\left(1+4x^2\right)^2} = \dfrac{4\left(1-4x^2\right)}{\left(1+4x^2\right)^2}$

For $x > 0$, $g'(x) = 0$ for $x = \frac{1}{2}$.

$g'(x) > 0$ for $0 < x < \frac{1}{2}$

$g'(x) < 0$ for $x > \frac{1}{2}$

$g\left(\frac{1}{2}\right) = 1$

Therefore g has a maximum value of 1 at $x = \frac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

$5 : \begin{cases} 2 : g'(x) \\ 1 : \text{critical point} \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

(b) $\displaystyle\int_1^{\infty} (f(x) - g(x))\,dx = \lim_{b\to\infty} \int_1^{b} (f(x) - g(x))\,dx$

$\displaystyle= \lim_{b\to\infty} \left(\ln(x) - \frac{1}{2}\ln\left(1+4x^2\right) \right)\Big|_{x=1}^{x=b}$

$\displaystyle= \lim_{b\to\infty} \left(\ln(b) - \frac{1}{2}\ln\left(1+4b^2\right) + \frac{1}{2}\ln(5) \right)$

$\displaystyle= \lim_{b\to\infty} \ln\left(\frac{b\sqrt{5}}{\sqrt{1+4b^2}} \right)$

$\displaystyle= \lim_{b\to\infty} \ln\left(\frac{\sqrt{5b^2}}{\sqrt{1+4b^2}} \right)$

$\displaystyle= \frac{1}{2} \lim_{b\to\infty} \ln\left(\frac{5b^2}{1+4b^2} \right)$

$\displaystyle= \frac{1}{2}\ln\frac{5}{4}$

$4 : \begin{cases} 1 : \text{integral} \\ 2 : \text{antidifferentiation} \\ 1 : \text{answer} \end{cases}$

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$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

6. The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

   (a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

   (c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

   (d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

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END OF EXAM

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Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

$f(x) = -\dfrac{1}{2} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} + \cdots + (-1)^{n+1} \dfrac{x^{2n}}{(2n+2)!} + \cdots$

3 : { 1 : terms for $\cos x$; 2 : terms for f { 1 : first three terms ; 1 : general term }

(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$\dfrac{f''(0)}{2!} = \dfrac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \dfrac{1}{12}$.

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

2 : { 1 : determines $f'(0)$; 1 : answer with reason }

(c) $P_5(x) = 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$

2 : { 1 : two correct terms ; 1 : remaining terms }

(d) $g(1) \approx 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} = \dfrac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \dfrac{37}{72} \right| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

2 : { 1 : estimate ; 1 : explanation }

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6. The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

   (a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

   (b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

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END OF EXAM

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Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = |2x|$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}$.

This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$.

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

5 : { 1 : sets up ratio; 1 : limit evaluation; 1 : radius of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

(b) $y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$

$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$

$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1} \cdot 2}{n-1} + \cdots$

$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$

$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n (2x)^n + \cdots$

$= 4x^2 \left(1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots\right)$

The series $1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots = \sum_{n=0}^{\infty} (-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$xy' - y = 4x^2 \cdot \frac{1}{1+2x}$ for $|x| < \frac{1}{2}$.

4 : { 1 : series for y'; 1 : series for xy'; 1 : series for $xy' - y$; 1 : analysis with geometric series }

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CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin\left(t^2\right)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

   (a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

   (b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

   (c) Find the position of the particle at time $t = 3$.

   (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

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Question 1

At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin\left(t^2\right)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

(a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 13.006$ or 13.007

Acceleration $= \langle x''(3), y''(3) \rangle$
$= \langle 4, -5.466 \rangle$ or $\langle 4, -5.467 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration} \end{cases}$

(b) Slope $= \frac{y'(3)}{x'(3)} = 0.031$ or 0.032

$1 : \text{answer}$

(c) $x(3) = 0 + \int_0^3 \frac{dx}{dt}\, dt = 21$

$y(3) = -4 + \int_0^3 \frac{dy}{dt}\, dt = -3.226$

At time $t = 3$, the particle is at position $(21, -3.226)$.

$4 : \begin{cases} 2 : x\text{-coordinate} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{answer} \\ 2 : y\text{-coordinate} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 21.091$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

   (a) According to the model, what is the height of the water in the can at the end of the 60-day period?

   (b) According to the model, what is the average rate of change in the height of water in the can over the 60-day period? Show the computations that lead to your answer. Indicate units of measure.

   (c) Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t = 7$? Indicate units of measure.

   (d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

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2011 SCORING GUIDELINES (Form B)

Question 1

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

(a) According to the model, what is the height of the water in the can at the end of the 60-day period?

(b) According to the model, what is the average rate of change in the height of water in the can over the 60-day period? Show the computations that lead to your answer. Indicate units of measure.

(c) Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t = 7$? Indicate units of measure.

(d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

(a) $S(60) = \int_0^{60} S'(t)\,dt = 171.813$ mm

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) $\frac{S(60) - S(0)}{60} = 2.863$ or 2.864 mm/day

1 : answer

(c) $V(t) = 100\pi S(t)$
$V'(7) = 100\pi S'(7) = 602.218$

The volume of water in the can is increasing at a rate of 602.218 mm^3/day.

2 : { 1 : relationship between V and S; 1 : answer }

(d) $D(0) = -0.675 < 0$ and $D(60) = 69.37730 > 0$

Because D is continuous, the Intermediate Value Theorem implies that there is a time t, $0 < t < 60$, at which $D(t) = 0$. At this time, the heights of water in the two cans are changing at the same rate.

2 : { 1 : considers $D(0)$ and $D(60)$; 1 : justification }

1 : units in (b) or (c)

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t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

   (d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

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END OF PART A OF SECTION II

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Question 2

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

(a) $H'(3.5) \approx \dfrac{H(5) - H(2)}{5 - 2}$

$= \dfrac{52 - 60}{3} = -2.666$ or -2.667 degrees Celsius per minute

1 : answer

(b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes.

$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2 \cdot \frac{66 + 60}{2} + 3 \cdot \frac{60 + 52}{2} + 4 \cdot \frac{52 + 44}{2} + 1 \cdot \frac{44 + 43}{2}\right)$

$= 52.95$

3 : { 1 : meaning of expression; 1 : trapezoidal sum; 1 : estimate }

(c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$

The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes.

2 : { 1 : value of integral; 1 : meaning of expression }

(d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275$; $\quad H(10) - B(10) = 8.817$

The biscuits are 8.817 degrees Celsius cooler than the tea.

3 : { 1 : integrand; 1 : uses $B(0) = 100$; 1 : answer }

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CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

2. The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

   (a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

   (b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

   (c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

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END OF PART A OF SECTION II

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2011 SCORING GUIDELINES (Form B)

Question 2

The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

(a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

(b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

(c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

(a) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} (r(\theta))^2 \, d\theta = 47.513$

3 : { 1 : integrand; 1 : limits and constant; 1 : answer }

(b) $-3 = r(\theta)\cos\theta = (3\theta + \sin\theta)\cos\theta$
$\theta = 2.01692$
$y = r(\theta)\sin(\theta) = 6.272$

3 : { 1 : equation; 1 : value of θ; 1 : y-coordinate }

(c) $y = r(\theta)\sin\theta = (3\theta + \sin\theta)\sin\theta$
$\left.\frac{dy}{dt}\right|_{\theta=2\pi/3} = \left[\frac{dy}{d\theta} \cdot \frac{d\theta}{dt}\right]_{\theta=2\pi/3} = -2.819$

The y-coordinate of the particle is decreasing at a rate of 2.819.

3 : { 1 : uses chain rule; 1 : answer; 1 : interpretation }

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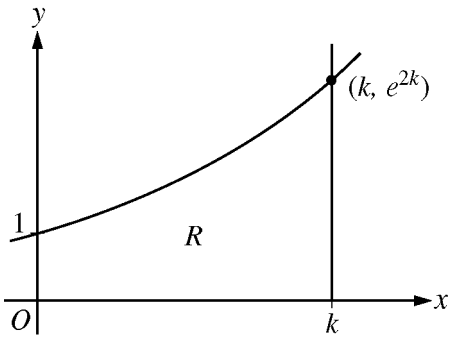
CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



3. Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.

   (a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

   (b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

   (c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

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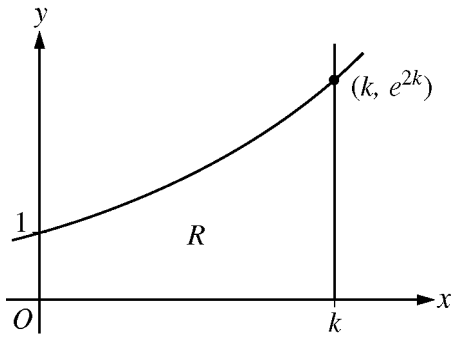
Question 3

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.



(a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

(b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

(c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

(a) $f'(x) = 2e^{2x}$

Perimeter $= 1 + k + e^{2k} + \int_0^k \sqrt{1 + \left(2e^{2x}\right)^2}\, dx$

3 : { 1 : $f'(x)$; 1 : integral; 1 : answer }

(b) Volume $= \pi\int_0^k \left(e^{2x}\right)^2 dx = \pi\int_0^k e^{4x}\, dx = \frac{\pi}{4}e^{4x}\Big|_{x=0}^{x=k} = \frac{\pi}{4}e^{4k} - \frac{\pi}{4}$

4 : { 1 : integrand; 1 : limits; 1 : antiderivative; 1 : answer }

(c) $\frac{dV}{dt} = \pi e^{4k}\frac{dk}{dt}$

When $k = \frac{1}{2}$, $\frac{dV}{dt} = \pi e^2 \cdot \frac{1}{3}$.

2 : { 1 : applies chain rule; 1 : answer }

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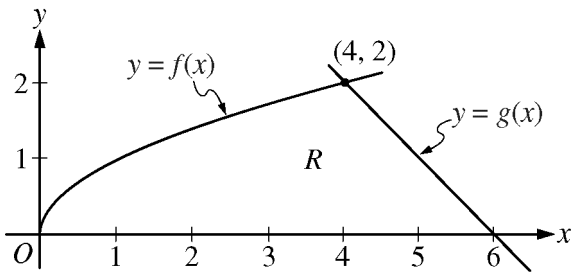
CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



3. The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

   (a) Find the area of R.

   (b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

   (c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.

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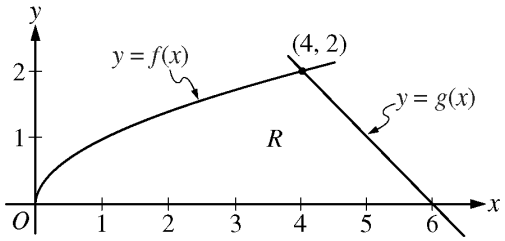
2011 SCORING GUIDELINES (Form B)

Question 3

The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.

(a) Area $= \int_0^4 \sqrt{x}\,dx + \frac{1}{2} \cdot 2 \cdot 2 = \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$

3 : { 1 : integral; 1 : antiderivative; 1 : answer }

(b) $y = \sqrt{x} \Rightarrow x = y^2$

$y = 6 - x \Rightarrow x = 6 - y$

Width $= (6 - y) - y^2$

Volume $= \int_0^2 2y\left(6 - y - y^2\right)dy$

3 : { 2 : integrand; 1 : answer }

(c) $g'(x) = -1$

Thus a line perpendicular to the graph of g has slope 1.

$f'(x) = \frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{x}} = 1 \Rightarrow x = \frac{1}{4}$

The point P has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

3 : { 1 : $f'(x)$; 1 : equation; 1 : answer }

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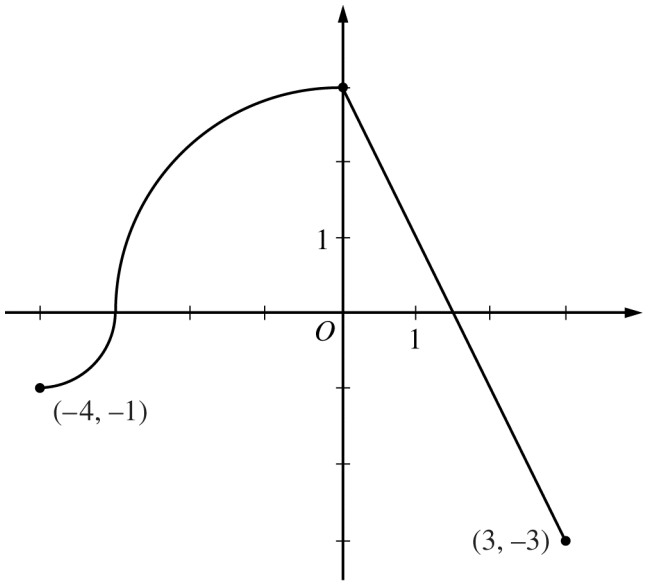

Graph of f

4. The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

   (a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

   (b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

   (c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

   (d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

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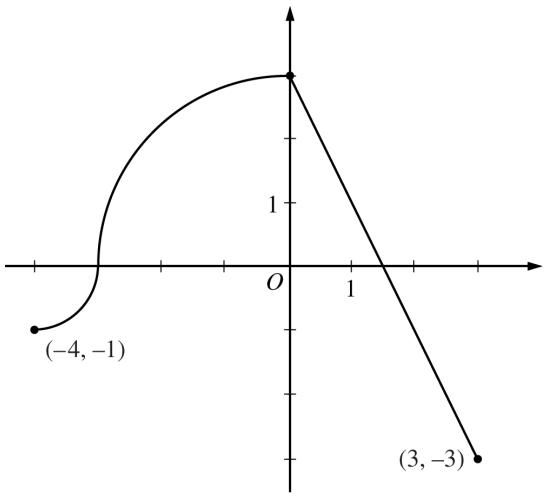
Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t)\,dt$.



Graph of f

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

3 : { 1 : $g(-3)$; 1 : $g'(x)$; 1 : $g'(-3)$ }

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.

$g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.

Therefore g has an absolute maximum at $x = \frac{5}{2}$.

3 : { 1 : considers $g'(x) = 0$; 1 : identifies interior candidate; 1 : answer with justification }

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.

1 : answer with reason

(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$.

To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.

2 : { 1 : average rate of change; 1 : explanation }

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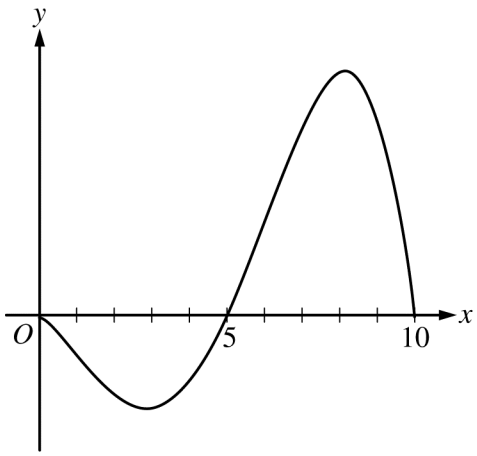

Graph of f

4. The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.

   (a) Find the average value of f on the interval $0 \le x \le 5$.

   (b) Evaluate $\int_0^{10} (3f(x) + 2)\, dx$. Show the computations that lead to your answer.

   (c) Let $g(x) = \int_5^x f(t)\, dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

   (d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

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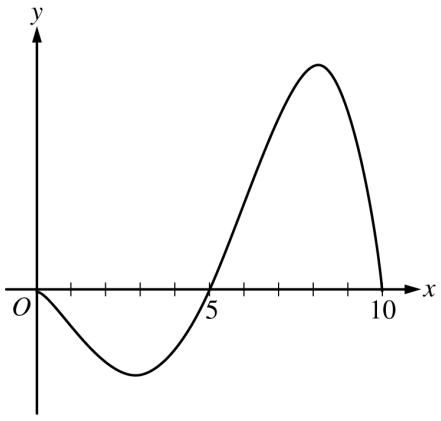
Question 4

The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.



Graph of f

(a) Find the average value of f on the interval $0 \le x \le 5$.

(b) Evaluate $\int_0^{10} (3f(x) + 2)\,dx$. Show the computations that lead to your answer.

(c) Let $g(x) = \int_5^x f(t)\,dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

(d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

(a) Average value $= \frac{1}{5}\int_0^5 f(x)\,dx = \frac{-10}{5} = -2$

1 : answer

(b) $\int_0^{10} (3f(x) + 2)\,dx = 3\left(\int_0^5 f(x)\,dx + \int_5^{10} f(x)\,dx\right) + 20$
$= 3(-10 + 27) + 20 = 71$

2 : answer

(c) $g'(x) = f(x)$
$g'(x) < 0$ on $0 < x < 5$
$g'(x)$ is increasing on $3 < x < 8$.
The graph of g is concave up and decreasing on $3 < x < 5$.

3 : { 1 : $g'(x) = f(x)$; 1 : analysis; 1 : answer and reason }

(d) Arc length $= \int_0^{20} \sqrt{1 + (h'(x))^2}\,dx = \int_0^{20} \sqrt{1 + \left(f'\left(\frac{x}{2}\right)\right)^2}\,dx$

Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}dx$ and
$\int_0^{20} \sqrt{1 + \left(f'\left(\frac{x}{2}\right)\right)^2}\,dx = 2\int_0^{10} \sqrt{1 + (f'(u))^2}\,du = 2(11 + 18) = 58$

3 : { 1 : integral; 1 : substitution; 1 : answer }

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5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

   (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

   (b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

   (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

2 : { 1 : $\frac{dW}{dt}$ at $t = 0$; 1 : answer }

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

2 : { 1 : $\frac{d^2W}{dt^2}$; 1 : answer with reason }

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \ 0 \leq t \leq 20$

5 : { 1 : separation of variables ; 1 : antiderivatives ; 1 : constant of integration ; 1 : uses initial condition ; 1 : solves for W }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

5. Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

   (a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

   (b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

   (c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

   (d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

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2011 SCORING GUIDELINES (Form B)

Question 5

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

(a) $a(5) \approx \dfrac{v(10) - v(0)}{10 - 0} = \dfrac{0.3}{10} = 0.03$ meters/sec^2

1 : answer

(b) $\int_0^{60} |v(t)|\,dt$ is the total distance, in meters, that Ben rides over the 60-second interval $t = 0$ to $t = 60$.

$\int_0^{60} |v(t)|\,dt \approx 2.0 \cdot 10 + 2.3(40 - 10) + 2.5(60 - 40) = 139$ meters

2 : { 1 : meaning of integral; 1 : approximation }

(c) Because $\dfrac{B(60) - B(40)}{60 - 40} = \dfrac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time t, $40 < t < 60$, such that $v(t) = 2$.

2 : { 1 : difference quotient; 1 : conclusion with justification }

(d) $2L(t)L'(t) = 2B(t)B'(t)$

$L'(40) = \dfrac{B(40)v(40)}{L(40)} = \dfrac{9 \cdot 2.5}{\sqrt{144 + 81}} = \dfrac{3}{2}$ meters/sec

3 : { 1 : derivatives; 1 : uses $B'(t) = v(t)$; 1 : answer }

1 : units in (a) or (b)

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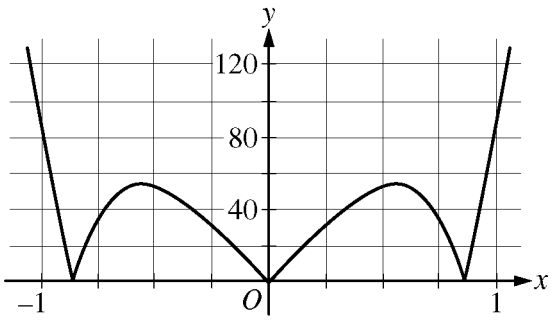

Graph of $y = \left|f^{(5)}(x)\right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

   (a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

   (b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

   (c) Find the value of $f^{(6)}(0)$.

   (d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

WRITE ALL WORK IN THE EXAM BOOKLET.

END OF EXAM

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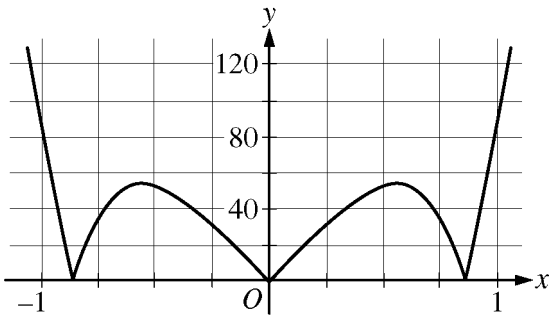
Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

3 : $\begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$

3 : $\begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

2 : $\begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

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6. Let $f(x) = \ln(1 + x^3)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

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END OF EXAM

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2011 SCORING GUIDELINES (Form B)

Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) The interval of convergence is centered at $x = 0$.

At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.

At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \le 1$.

2 : answer with analysis

(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are

$$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$$

$$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$$

Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.

4 : $\begin{cases} 1 : \text{two terms for } f'\left(t^2\right) \\ 1 : \text{other terms for } f'\left(t^2\right) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.

Thus $\left|g(1) - \frac{18}{55}\right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.

1 : analysis

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CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

   (a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

   (c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

   (d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

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Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is $55°$F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6}$
$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately $1.017\,°$F per minute at time $t = 12$ minutes.

2 : { 1 : estimate; 1 : interpretation with units }

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by $16\,°$F over the interval from $t = 0$ to $t = 20$ minutes.

2 : { 1 : value; 1 : interpretation with units }

(c) $\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$
$= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$
$= \frac{1}{20} \cdot 1215.8 = 60.79$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

3 : { 1 : left Riemann sum; 1 : approximation; 1 : underestimate with reason }

(d) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$
$= 71.0 + 2.043155 = 73.043$

2 : { 1 : integral; 1 : answer }

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2. For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

END OF PART A OF SECTION II

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Question 2

For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

(a) $\left.\frac{dx}{dt}\right|_{t=2} = \frac{2}{e^2}$

Because $\left.\frac{dx}{dt}\right|_{t=2} > 0$, the particle is moving to the right at time $t = 2$.

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{dy/dt|_{t=2}}{dx/dt|_{t=2}} = 3.055$ (or 3.054)

3 : { 1 : moving to the right with reason; 1 : considers $\frac{dy/dt}{dx/dt}$; 1 : slope at $t = 2$ }

(b) $x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t}\,dt = 1.253$ (or 1.252)

2 : { 1 : integral; 1 : answer }

(c) Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = 0.575$ (or 0.574)

Acceleration $= \langle x''(4), y''(4) \rangle$
$= \langle -0.041, 0.989 \rangle$

2 : { 1 : speed; 1 : acceleration }

(d) Distance $= \int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
$= 0.651$ (or 0.650)

2 : { 1 : integral; 1 : answer }

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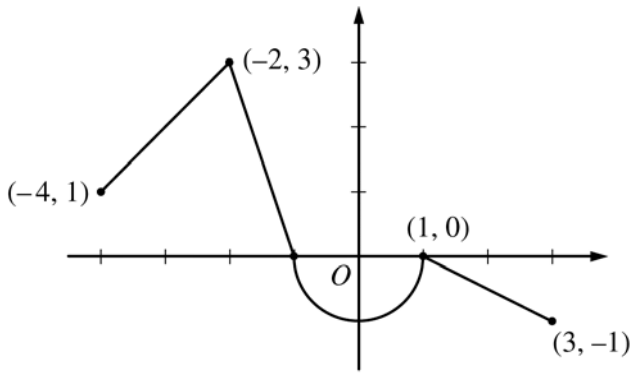
CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

   (a) Find the values of $g(2)$ and $g(-2)$.

   (b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

   (c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

   (d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

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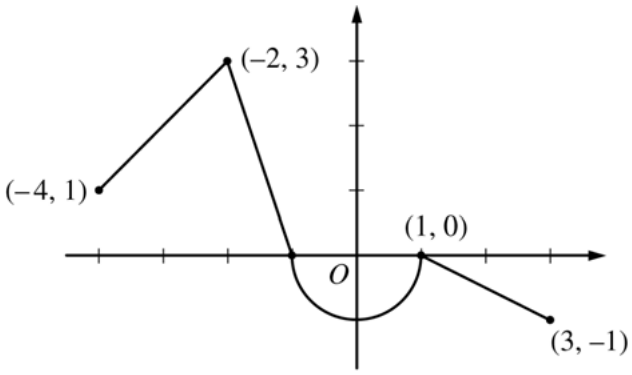
Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$

$g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

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x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

4. The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

   (a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

   (b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

   (c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

   (d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15, \quad f'(1) = 8$

An equation for the tangent line is $y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3 : \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$

$= 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$2 : \begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$

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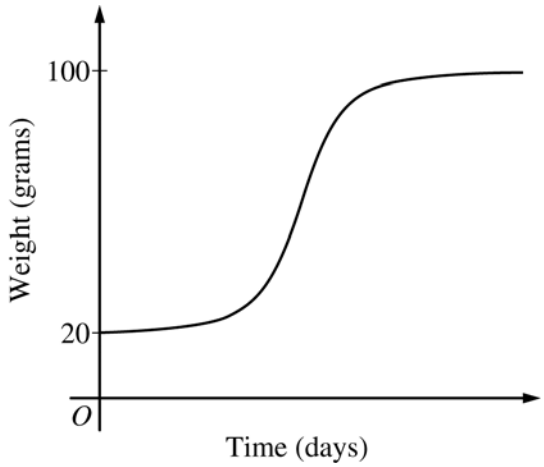
5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

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Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

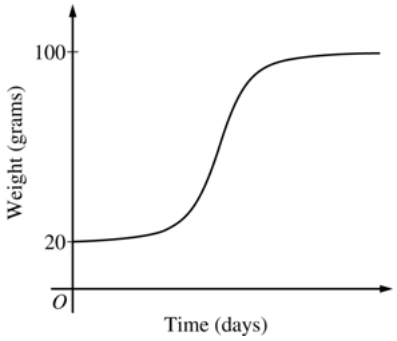
$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

2 : { 1 : uses $\frac{dB}{dt}$; 1 : answer with reason }

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

2 : { 1 : $\frac{d^2B}{dt^2}$ in terms of B ; 1 : explanation }

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \;\Rightarrow\; -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

5 : { 1 : separation of variables ; 1 : antiderivatives ; 1 : constant of integration ; 1 : uses initial condition ; 1 : solves for B }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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6. The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

   (a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

   (b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

   (c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

STOP

END OF EXAM

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Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : identifies interior of interval of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 : { 1 : uses the third term as an error bound; 1 : error bound }

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 : { 1 : first three terms; 1 : general term }

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CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

   (a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

   (b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

   (c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

   (d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

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Question 1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

(a) $G'(5) = -24.588$ (or -24.587)

The rate at which gravel is arriving is decreasing by 24.588 (or 24.587) tons per hour per hour at time $t = 5$ hours.

$2 : \begin{cases} 1 : G'(5) \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\int_0^8 G(t)\,dt = 825.551$ tons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $G(5) = 98.140764 < 100$

At time $t = 5$, the rate at which unprocessed gravel is arriving is less than the rate at which it is being processed. Therefore, the amount of unprocessed gravel at the plant is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{compares } G(5) \text{ to } 100 \\ 1 : \text{conclusion} \end{cases}$

(d) The amount of unprocessed gravel at time t is given by $A(t) = 500 + \int_0^t (G(s) - 100)\,ds$.

$A'(t) = G(t) - 100 = 0 \Rightarrow t = 4.923480$

t	$A(t)$
0	500
4.92348	635.376123
8	525.551089

The maximum amount of unprocessed gravel at the plant during this workday is 635.376 tons.

$3 : \begin{cases} 1 : \text{considers } A'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

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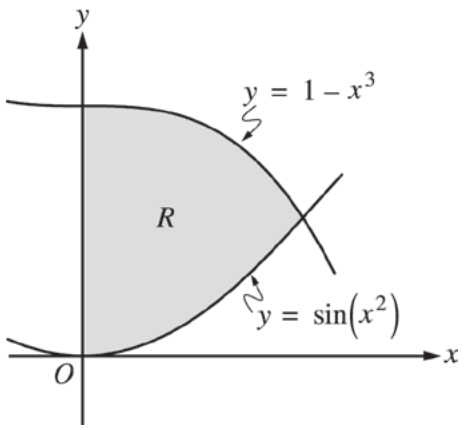
A graphing calculator is required for these problems.



1. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$, as shown in the figure above.

   (a) Find the area of R.

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Continue problem 1 on page 5.

A graphing calculator is required for these problems.

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion.

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

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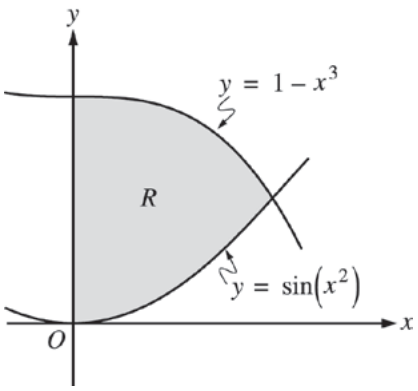
Question 1

Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$, as shown in the figure above.



(a) Find the area of R.

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion.

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

The graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$ intersect in the first quadrant at the point $(A, B) = (0.764972, 0.552352)$.

1 : correct limits in an integral in (a), (b), or (c)

(a) $\text{Area} = \int_0^A \left(1 - x^3 - \sin\left(x^2\right)\right) dx$
$= 0.533$ (or 0.534)

2 : { 1 : integrand; 1 : answer }

(b) $k = B = 0.552352$

$\int_0^A \left(1 - x^3 - k\right) dx = 0.257$ (or 0.256)

$\int_0^A \left(k - \sin\left(x^2\right)\right) dx = 0.277$ (or 0.276)

The two regions have unequal areas.

3 : { 1 : integral(s) with k value; 1 : value(s) of integral(s); 1 : conclusion tied to part (a) or comparison of two integrals }

Note: Stating k value only does not earn a point.

(c) $\text{Volume} = \pi \int_0^A \left(\left(1 - x^3 + 3\right)^2 - \left(\sin\left(x^2\right) + 3\right)^2\right) dx$
$= 11.841$ (or 11.840)

3 : { 2 : integrand; 1 : answer }

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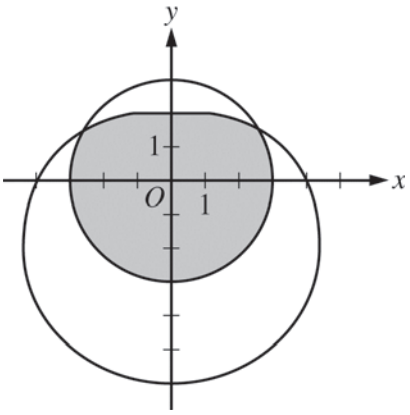

2. The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

   (a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

   (b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

   (c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

END OF PART A OF SECTION II

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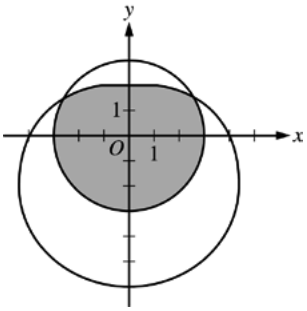
Question 2

The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.



(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

(a) Area $= 6\pi + \frac{1}{2}\int_{\pi/6}^{5\pi/6}(4 - 2\sin\theta)^2\, d\theta = 24.709$ (or 24.708)

3 : { 1 : integrand; 1 : limits and constant; 1 : answer }

(b) $x = r\cos\theta \Rightarrow x(\theta) = (4 - 2\sin\theta)\cos\theta$

$x(t) = \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right)$

$x(t) = -1$ when $t = 1.428$ (or 1.427)

3 : { 1 : $x(\theta)$ or $x(t)$; 1 : $x(\theta) = -1$ or $x(t) = -1$; 1 : answer }

(c) $y = r\sin\theta \Rightarrow y(\theta) = (4 - 2\sin\theta)\sin\theta$

$y(t) = \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)$

Position vector $= \langle x(t), y(t)\rangle$

$= \left\langle \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right), \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)\right\rangle$

$v(1.5) = \langle x'(1.5), y'(1.5)\rangle$

$= \langle -8.072, -1.673\rangle$ (or $\langle -8.072, -1.672\rangle$)

3 : { 2 : position vector; 1 : velocity vector }

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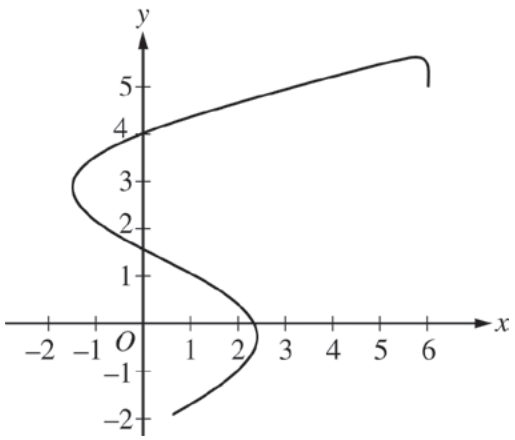
A graphing calculator is required for these problems.



2. A planetary rover travels on a flat surface. The path of the rover for the time interval $0 \le t \le 2$ hours is shown in the rectangular coordinate system above. The rover starts at the point with coordinates $(6, 5)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position of the rover change at rates given by

$$x'(t) = -12\sin\left(2t^2\right)$$
$$y'(t) = 10\cos\left(1 + \sqrt{t}\right),$$

where $x(t)$ and $y(t)$ are measured in meters and t is measured in hours.

(a) Find the acceleration vector of the rover at time $t = 1$. Find the speed of the rover at time $t = 1$.

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Continue problem 2 on page 7.

A graphing calculator is required for these problems.

(b) Find the total distance that the rover travels over the time interval $0 \le t \le 1$.

(c) Find the y-coordinate of the position of the rover at time $t = 1$.

(d) The rover receives a signal at each point where the line tangent to its path has slope $\frac{1}{2}$. At what times t, for $0 \le t \le 2$, does the rover receive a signal?

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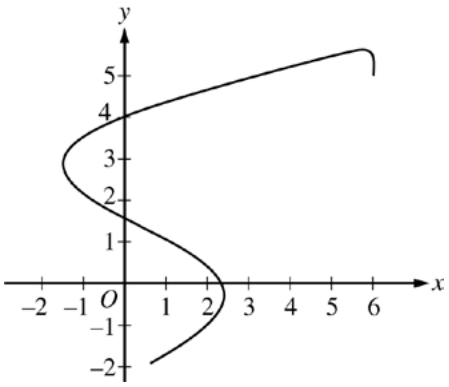
Question 2

A planetary rover travels on a flat surface. The path of the rover for the time interval $0 \le t \le 2$ hours is shown in the rectangular coordinate system above. The rover starts at the point with coordinates $(6, 5)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position of the rover change at rates given by

$$x'(t) = -12\sin\left(2t^2\right)$$
$$y'(t) = 10\cos\left(1 + \sqrt{t}\right),$$



where $x(t)$ and $y(t)$ are measured in meters and t is measured in hours.

(a) Find the acceleration vector of the rover at time $t = 1$. Find the speed of the rover at time $t = 1$.

(b) Find the total distance that the rover travels over the time interval $0 \le t \le 1$.

(c) Find the y-coordinate of the position of the rover at time $t = 1$.

(d) The rover receives a signal at each point where the line tangent to its path has slope $\frac{1}{2}$. At what times t, for $0 \le t \le 2$, does the rover receive a signal?

(a) $a(1) = \langle x''(1), y''(1) \rangle \quad \langle 19.975, -4.546 \rangle$

Speed $= \sqrt{(x'(1))^2 + (y'(1))^2} = 11.678$

2 : { 1 : acceleration vector; 1 : speed }

(b) Distance $= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 6.704$ (or 6.703)

3 : { 2 : integral; 1 : answer }

(c) $y(1) = 5 + \int_0^1 y'(t)\, dt = 4.057$ (or 4.056)

2 : { 1 : integral; 1 : answer }

(d) $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{10\cos\left(1 + \sqrt{t}\right)}{-12\sin\left(2t^2\right)} = \dfrac{1}{2}$

$t = 1.072$

2 : { 1 : equation; 1 : answer }

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CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

   (a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

   (b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

   (c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

   (d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

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Question 3

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

(a) $C'(3.5) \approx \dfrac{C(4) - C(3)}{4 - 3} = \dfrac{12.8 - 11.2}{1} = 1.6$ ounces/min

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{units} \end{cases}$

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval)

$\dfrac{C(4) - C(2)}{4 - 2} = \dfrac{12.8 - 8.8}{2} = 2$

Therefore, by the Mean Value Theorem, there is at least one time t, $2 < t < 4$, for which $C'(t) = 2$.

$2 : \begin{cases} 1 : \dfrac{C(4) - C(2)}{4 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c) $\dfrac{1}{6}\int_0^6 C(t)\,dt \approx \dfrac{1}{6}[2 \cdot C(1) + 2 \cdot C(3) + 2 \cdot C(5)]$

$= \dfrac{1}{6}(2 \cdot 5.3 + 2 \cdot 11.2 + 2 \cdot 13.8)$

$= \dfrac{1}{6}(60.6) = 10.1$ ounces

$\dfrac{1}{6}\int_0^6 C(t)\,dt$ is the average amount of coffee in the cup, in ounces, over the time interval $0 \le t \le 6$ minutes.

$3 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{approximation} \\ 1 : \text{interpretation} \end{cases}$

(d) $B'(t) = -16(-0.4)e^{-0.4t} = 6.4e^{-0.4t}$

$B'(5) = 6.4e^{-0.4(5)} = \dfrac{6.4}{e^2}$ ounces/min

$2 : \begin{cases} 1 : B'(t) \\ 1 : B'(5) \end{cases}$

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t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

3. The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in gigaliters (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

   (a) Use the tangent line approximation to W at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

   (b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate $\int_0^{30} W'(t)\,dt$. Use this approximation to estimate the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 0$. Show the computations that lead to your answer.

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Continue problem 3 on page 15.

(c) Explain why there must be at least one time t, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometers per day, with respect to t when $t = 30$ days.

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Question 3

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in gigaliters (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

(a) Use the tangent line approximation to W at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

(b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate $\int_0^{30} W'(t)\,dt$. Use this approximation to estimate the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 0$. Show the computations that lead to your answer.

(c) Explain why there must be at least one time t, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometers per day, with respect to t when $t = 30$ days.

(a) An equation of the tangent line is $y = 0.5(t - 30) + 125$.
$W(32) \approx 0.5(32 - 30) + 125 = 126$

1 : answer

(b) $\int_0^{30} W'(t)\,dt \approx (10)(0.6) + (12)(0.7) + (8)(1.0) = 22.4$

$W(0) = W(30) - \int_0^{30} W'(t)\,dt = 125 - 22.4 = 102.6$

3 : 1 : left Riemann sum; 1 : approximation; 1 : answer

(c) W' is differentiable $\Rightarrow$ W' is continuous.

$W'(30) = 0.5 < 0.7 < 1.0 = W'(22)$

By the Intermediate Value Theorem, there must be at least one time t, $22 \le t \le 30$, such that $W'(t) = 0.7$.

2 : explanation

(d) $\dfrac{dA}{dt} = (0.3)\dfrac{2}{3}W^{-1/3} \cdot \dfrac{dW}{dt} = \dfrac{0.2}{\sqrt[3]{W}} \cdot \dfrac{dW}{dt}$

$\left.\dfrac{dA}{dt}\right|_{t=30} = \dfrac{0.2}{\sqrt[3]{125}} \cdot 0.5 = 0.02$

3 : 2 : $\dfrac{dA}{dt}$; 1 : answer

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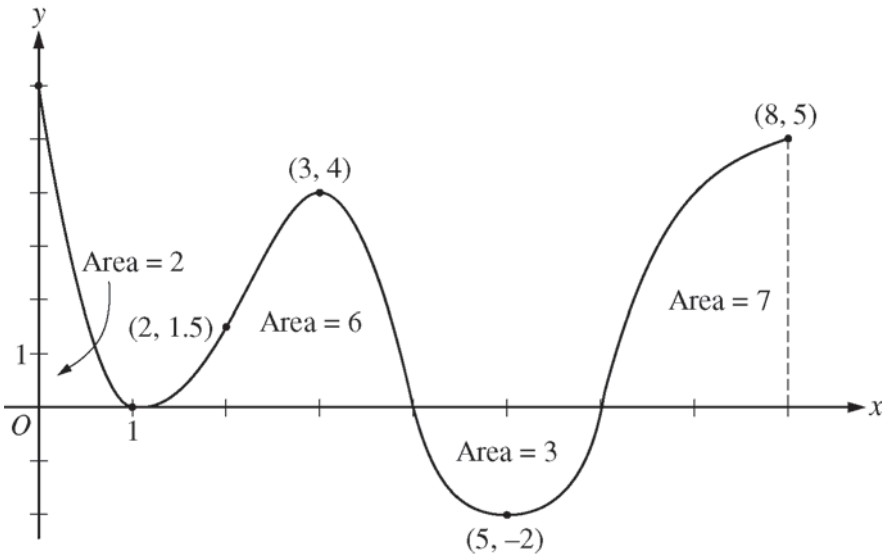

Graph of f'

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

   (a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

   (b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

   (c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

   (d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

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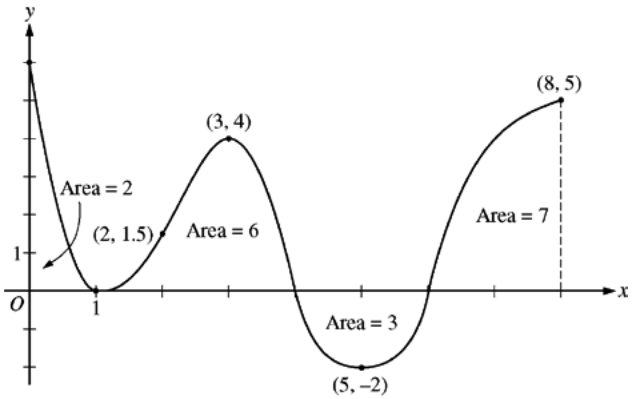
Question 4

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

1 : answer with justification

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$f(0) = f(8) + \int_8^0 f'(x)\,dx$$
$$= f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$$
$$f(6) = f(8) + \int_8^6 f'(x)\,dx$$
$$= f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$
$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

3 : { 1 : considers $x = 0$ and $x = 6$; 1 : answer; 1 : justification }

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

2 : { 1 : answer; 1 : explanation }

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$

3 : { 2 : $g'(x)$; 1 : answer }

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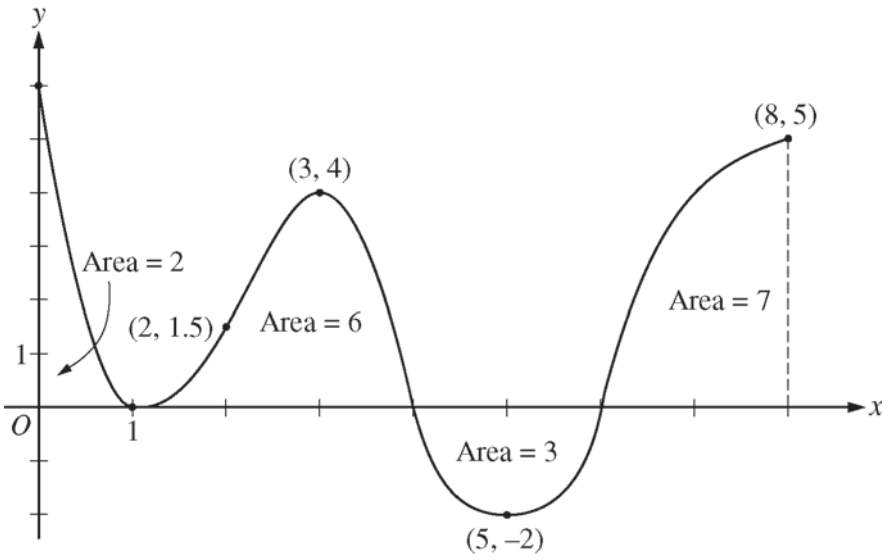

Graph of f'

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Question 4

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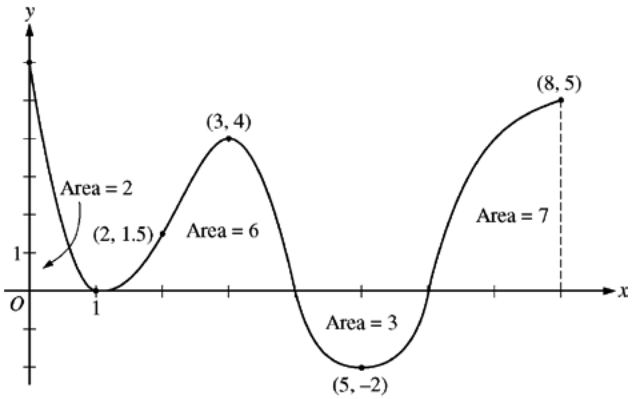

Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

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$$= f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$
$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

3 : { 1 : considers $x = 0$ and $x = 6$; 1 : answer; 1 : justification }

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

2 : { 1 : answer; 1 : explanation }

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$

3 : { 2 : $g'(x)$; 1 : answer }

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4. Consider the function f given by $f(x) = xe^{-x^2}$ for all real numbers x.

(a) At what value of x does $f(x)$ attain its absolute maximum? Justify your answer.

(b) Find an antiderivative of f.

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(c) Find the value of $\int_0^\infty x f(x)\,dx$, given the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

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Question 4

$f(x) = xe^{-x^2}$ for all real numbers x.

(a) At what value of x does $f(x)$ attain its absolute maximum? Justify your answer.

(b) Find an antiderivative of f.

(c) Find the value of $\int_0^\infty x f(x)\,dx$, given the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

(a) $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = \left(1 - 2x^2\right)e^{-x^2}$

$f'(x) = 0$ when $x = -\frac{1}{\sqrt{2}}$ or $x = \frac{1}{\sqrt{2}}$

$f(x) < 0$ for $x < 0$, and $f(x) > 0$ for $x > 0$. Therefore, a maximum can only occur at a positive x-value.

$f'(x)$ changes sign from positive to negative at $x = \frac{1}{\sqrt{2}}$, so

f has a relative maximum at $x = \frac{1}{\sqrt{2}}$.

f increases on $0 < x \le \frac{1}{\sqrt{2}}$ and decreases on $\frac{1}{\sqrt{2}} \le x < \infty$.

Therefore, there is an absolute maximum at $x = \frac{1}{\sqrt{2}}$.

5 : { 2 : $f'(x)$; 1 : solves $f'(x) = 0$; 1 : answer; 1 : justification }

(b) $\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C$

1 : antiderivative

(c) Using integration by parts,

$u = x \quad dv = xe^{-x^2}\,dx$

$du = dx \quad v = -\frac{1}{2}e^{-x^2}$

$\int x f(x)\,dx = \int x^2e^{-x^2}\,dx = -\frac{1}{2}xe^{-x^2} + \frac{1}{2}\int e^{-x^2}\,dx$

Therefore,

$$\int_0^\infty x f(x)\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}xe^{-x^2}\Big|_0^b + \frac{1}{2}\int_0^b e^{-x^2}\,dx\right]$$
$$= \lim_{b\to\infty}\left(-\frac{1}{2}be^{-b^2}\right) + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$
$$= 0 + \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

3 : { 2 : integration by parts; 1 : answer }

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5. Consider the differential equation $\frac{dy}{dx} = y^2(2x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

   (a) Find $\lim_{x \to 0} \frac{f(x) + 1}{\sin x}$. Show the work that leads to your answer.

   (b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

   (c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

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Question 5

Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0} \sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0} \frac{f(x)+1}{\sin x} = \lim_{x\to 0} \frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2 \cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b) $f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right)$

$$= -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$

$$= -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = y^2(2x+2)$

$$\frac{dy}{y^2} = (2x+2)\,dx$$

$$\int \frac{dy}{y^2} = \int (2x+2)\,dx$$

$$-\frac{1}{y} = x^2 + 2x + C$$

$$-\frac{1}{-1} = 0^2 + 2\cdot 0 + C \Rightarrow C = 1$$

$$-\frac{1}{y} = x^2 + 2x + 1$$

$$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$$

Note: This solution is valid for $x > -1$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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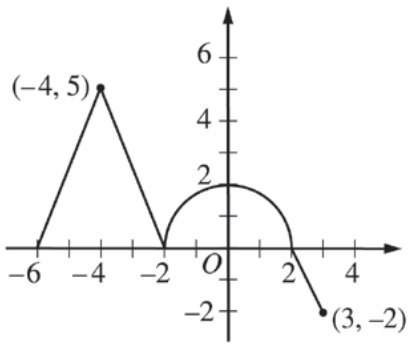

Graph of f

5. The graph of the continuous function f, consisting of three line segments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(-6)$ and $g(3)$.

(b) Find $g'(0)$.

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(c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

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Question 5

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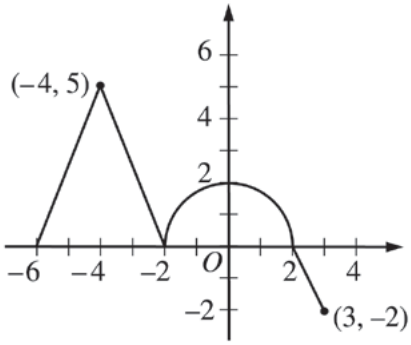

Graph of f

(a) Find $g(-6)$ and $g(3)$.

(b) Find $g'(0)$.

(c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(-6) = \int_{-2}^{-6} f(t)\,dt = -\int_{-6}^{-2} f(t)\,dt = -\frac{1}{2}\cdot 4\cdot 5 = -10$

$g(3) = \int_{-2}^{3} f(t)\,dt = \frac{1}{2}\pi\cdot 2^2 - \frac{1}{2}\cdot 1\cdot 2 = 2\pi - 1$

$2 : \begin{cases} 1 : g(-6) \\ 1 : g(3) \end{cases}$

(b) $g'(0) = f(0) = 2$

$1 : g'(0)$

(c) The graph of g has a horizontal tangent at $x = -2$ and $x = 2$ where $g'(x) = f(x) = 0$.

The graph of g has neither a local maximum nor a local minimum at $x = -2$ because $g'(x) = f(x)$ does not change sign at $x = -2$.

The graph of g has a local maximum at $x = 2$ because $g'(x) = f(x)$ changes sign from positive to negative at $x = 2$.

$3 : \begin{cases} 1 : \text{horizontal tangent at } x = -2 \text{ and } x = 2 \\ 2 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at $x = -4$, $x = -2$, and $x = 0$.
$g'(x) = f(x)$ changes from increasing to decreasing at $x = -4$ and $x = 0$, and changes from decreasing to increasing at $x = -2$.

OR

$g''(x) = f'(x)$ changes from positive to negative at $x = -4$ and $x = 0$, and changes from negative to positive at $x = -2$.

$3 : \begin{cases} 2 : \text{values of } x \\ 1 : \text{explanation} \end{cases}$

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6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

   (a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

   (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

   (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

STOP

END OF EXAM

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Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$P_1\left(\frac{1}{2}\right) = -4 + f'(0)\cdot\frac{1}{2} = -3$

$f'(0)\cdot\frac{1}{2} = 1$

$f'(0) = 2$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right)\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{x^3}{3!}$

$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$

$Q_3(x) = -4x + 4\cdot\frac{x^2}{2} - \frac{4}{3}\cdot\frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$

$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

OR

$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$

$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$

$Q_3(x) = 7 - 4x + 4\cdot\frac{x^2}{2!} - \frac{8}{3}\cdot\frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

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6. The function f satisfies the equation

$$f'(x) = f(x) + x + 1$$

and $f(0) = 2$. The Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

(a) Write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 0$.

(b) Find $f''(0)$ and find the second-degree Taylor polynomial for f about $x = 0$.

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(c) Find the fourth-degree Taylor polynomial for f about $x = 0$.

(d) Find $f^{(n)}(0)$, the nth derivative of f at $x = 0$, for $n \geq 2$. Use the Taylor series for f about $x = 0$ and the Taylor series for e^x about $x = 0$ to find a polynomial expression for $f(x) - 4e^x$.

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Question 6

The function f satisfies the equation

$$f'(x) = f(x) + x + 1$$

and $f(0) = 2$. The Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

(a) Write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 0$.

(b) Find $f''(0)$ and find the second-degree Taylor polynomial for f about $x = 0$.

(c) Find the fourth-degree Taylor polynomial for f about $x = 0$.

(d) Find $f^{(n)}(0)$, the nth derivative of f at $x = 0$, for $n \geq 2$. Use the Taylor series for f about $x = 0$ and the Taylor series for e^x about $x = 0$ to find a polynomial expression for $f(x) - 4e^x$.

(a) $f'(0) = f(0) + 0 + 1 = 3$
An equation for the tangent line is $y = 3x + 2$.

1 : tangent line equation

(b) $f''(x) = f'(x) + 1;\ f''(0) = f'(0) + 1 = 3 + 1 = 4$

$P_2(x) = 2 + 3x + \frac{4}{2!}x^2 = 2 + 3x + 2x^2$

2 : { 1 : $f''(0)$; 1 : second-degree Taylor polynomial }

(c) $f'''(x) = f''(x);\ f'''(0) = f''(0) = 4$

$f^{(4)}(x) = f'''(x);\ f^{(4)}(0) = f'''(0) = 4$

$P_4(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4$
$= 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$

2 : { 1 : $f'''(0)$ and $f^{(4)}(0)$; 1 : fourth-degree Taylor polynomial }

(d) $f^{(n)}(0) = 4$ for $n \geq 2$

$f(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \frac{4}{5!}x^5 + \cdots$

$4e^x = 4 + 4x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \frac{4}{5!}x^5 + \cdots$

Therefore, $f(x) - 4e^x = -2 - x$.

4 : { 1 : $f^{(n)}(0)$ for $n \geq 2$; 1 : Taylor series for f; 1 : Taylor series for e^x; 1 : polynomial expression }

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CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

   (b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

   (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

   (d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

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Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

(a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or -0.196) lbs/day

1 : answer with units

(b) $A'(15) = -0.164$ (or -0.163)

The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.

$2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$

(c) $A(t) = \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414)

$2 : \begin{cases} 1 : \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$

(d) $L(t) = A(30) + A'(30) \cdot (t - 30)$

$A'(30) = -0.055976$

$A(30) = 0.782928$

$L(t) = 0.5 \Rightarrow t = 35.054$

$4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

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A graphing calculator is required for these problems.

t (hours)	0	0.4	0.8	1.2	1.6	2.0	2.4
$v(t)$ (miles per hour)	0	11.8	9.5	17.2	16.3	16.8	20.1

1. Ruth rode her bicycle on a straight trail. She recorded her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 2.4$ hours, as shown in the table above. For $0 < t \le 2.4$, $v(t) > 0$.

   (a) Use the data in the table to approximate Ruth's acceleration at time $t = 1.4$ hours. Show the computations that lead to your answer. Indicate units of measure.

   (b) Using correct units, interpret the meaning of $\int_0^{2.4} v(t)\,dt$ in the context of the problem. Approximate $\int_0^{2.4} v(t)\,dt$ using a midpoint Riemann sum with three subintervals of equal length and values from the table.

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Continue problem 1 on page 5.

A graphing calculator is required for these problems.

(c) For $0 \le t \le 2.4$ hours, Ruth's velocity can be modeled by the function g given by $g(t) = \dfrac{24t + 5\sin(6t)}{t + 0.7}$. According to the model, what was Ruth's average velocity during the time interval $0 \le t \le 2.4$?

(d) According to the model given in part (c), is Ruth's speed increasing or decreasing at time $t = 1.3$? Give a reason for your answer.

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Question 1

t (hours)	0	0.4	0.8	1.2	1.6	2.0	2.4
$v(t)$ (miles per hour)	0	11.8	9.5	17.2	16.3	16.8	20.1

Ruth rode her bicycle on a straight trail. She recorded her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 2.4$ hours, as shown in the table above. For $0 < t \le 2.4$, $v(t) > 0$.

(a) Use the data in the table to approximate Ruth's acceleration at time $t = 1.4$ hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{2.4} v(t)\,dt$ in the context of the problem. Approximate $\int_0^{2.4} v(t)\,dt$ using a midpoint Riemann sum with three subintervals of equal length and values from the table.

(c) For $0 \le t \le 2.4$ hours, Ruth's velocity can be modeled by the function g given by $g(t) = \dfrac{24t + 5\sin(6t)}{t + 0.7}$. According to the model, what was Ruth's average velocity during the time interval $0 \le t \le 2.4$?

(d) According to the model given in part (c), is Ruth's speed increasing or decreasing at time $t = 1.3$? Give a reason for your answer.

(a) $a(1.4) \approx \dfrac{v(1.6) - v(1.2)}{1.6 - 1.2} = \dfrac{16.3 - 17.2}{1.6 - 1.2} = -2.25$ miles/hr^2

2 : { 1 : approximation; 1 : units }

(b) $\int_0^{2.4} v(t)\,dt$ is the total distance Ruth traveled, in miles, from time $t = 0$ to time $t = 2.4$ hours.

$\int_0^{2.4} v(t)\,dt \approx (0.8)(11.8) + (0.8)(17.2) + (0.8)(16.8)$
$= 36.64$ miles

3 : { 1 : interpretation; 1 : midpoint Riemann sum; 1 : approximation }

(c) Average velocity $= \dfrac{1}{2.4}\int_0^{2.4} g(t)\,dt = 14.064$ miles/hr

2 : { 1 : integral; 1 : answer }

(d) Velocity $= g(1.3) = 18.096358 > 0$
Acceleration $= g'(1.3) = 3.761152 > 0$

Ruth's speed is increasing at time $t = 1.3$ since velocity and acceleration have the same sign at this time.

2 : conclusion with reason

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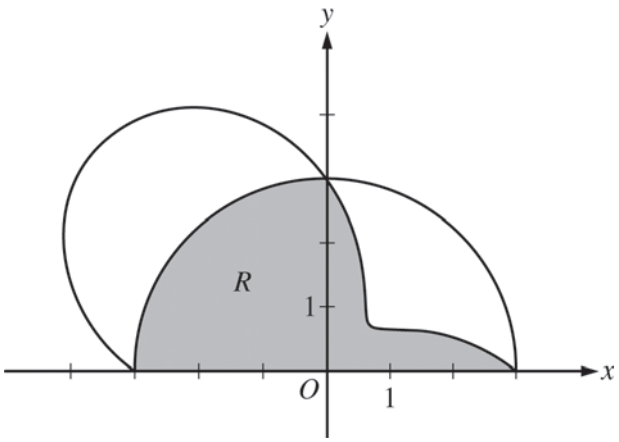

2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

   (a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

   (b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

   (c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

   (d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

END OF PART A OF SECTION II

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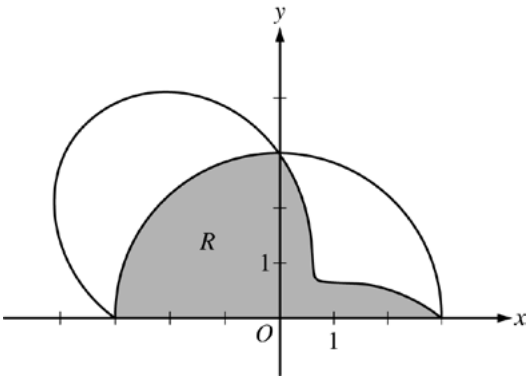
Question 2

The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.



(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a) $\text{Area} = \frac{9\pi}{4} + \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta$
$= 9.708$ (or 9.707)

3 : { 1 : integrand; 1 : limits; 1 : answer }

(b) $x = (3 - 2\sin(2\theta))\cos\theta$
$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -2.366$

2 : { 1 : expression for x; 1 : answer }

(c) The distance between the two curves is $D = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$.
$\left.\frac{dD}{d\theta}\right|_{\theta=\pi/3} = -2$

2 : { 1 : expression for distance; 1 : answer }

(d) $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot 3$
$\left.\frac{dr}{dt}\right|_{\theta=\pi/6} = (-2)(3) = -6$

2 : { 1 : chain rule with respect to t; 1 : answer }

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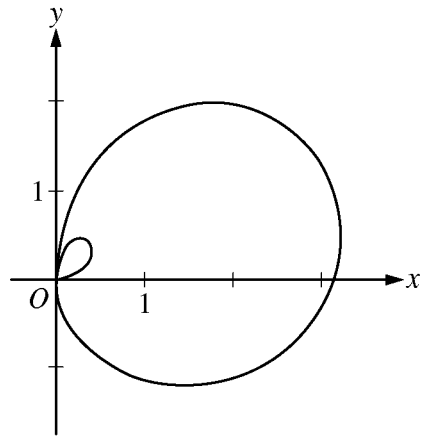
A graphing calculator is required for these problems.



2. Consider the polar curve defined by the function $r(\theta) = \theta\cos\theta$, where $0 \le \theta \le \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = \cos\theta - \theta\sin\theta$. The figure above shows the graph of r for $0 \le \theta \le \frac{3\pi}{2}$.

   (a) Find the area of the region enclosed by the inner loop of the curve.

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Continue problem 2 on page 7.

A graphing calculator is required for these problems.

(b) For $0 \le \theta \le \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

(c) There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

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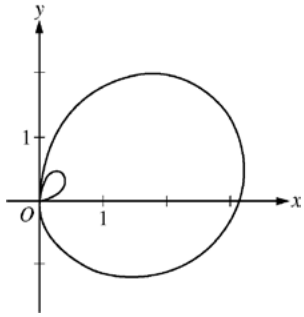
Question 2

Consider the polar curve defined by the function $r(\theta) = \theta\cos\theta,$ where $0 \le \theta \le \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = \cos\theta - \theta\sin\theta$. The figure above shows the graph of r for $0 \le \theta \le \frac{3\pi}{2}$.



(a) Find the area of the region enclosed by the inner loop of the curve.

(b) For $0 \le \theta \le \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

(c) There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

(a) Area $= \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2\, d\theta = 0.127$ (or 0.126)

3 : { 1 : integrand; 1 : limits and constant; 1 : answer }

(b) $r'(\theta) = \cos\theta - \theta\sin\theta = 0 \Rightarrow \theta = 0.860334,\ 3.425618$

θ	$r(\theta)$
0	0
0.860334	0.561096
3.425618	–3.288371
$\frac{3\pi}{2}$	0

Therefore, the greatest distance from any point on the graph of r to the origin is 3.288.

3 : { 1 : identifies $\theta = 3.425618$ as a candidate; 1 : answer; 1 : justification }

(c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

At the point where the tangent line has slope $\frac{2}{2-\pi}$,

$\frac{2}{2-\pi} = \frac{1/2}{dx/d\theta}$.

Therefore, $\frac{dx}{d\theta} = \frac{1}{2}\cdot\frac{2-\pi}{2} = \frac{2-\pi}{4}$ at this point.

3 : { 1 : $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$; 1 : equation; 1 : answer }

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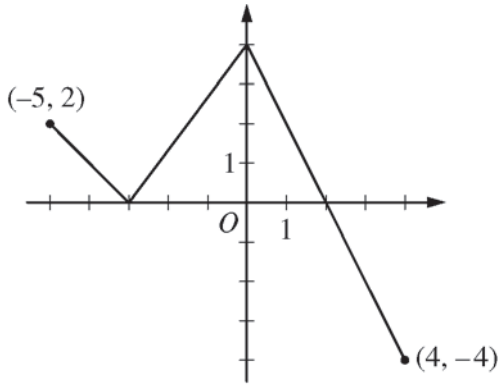
CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

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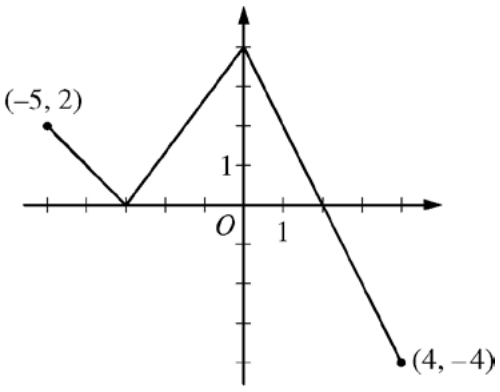
Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.



Graph of f

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^{3} f(t)\,dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

2 : { 1 : answer; 1 : reason }

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$h'(3) = \dfrac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$

$= \dfrac{15(-2) - 5(9)}{225} = \dfrac{-75}{225} = -\dfrac{1}{3}$

3 : { 2 : $h'(x)$; 1 : answer }

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

3 : { 2 : $p'(x)$; 1 : answer }

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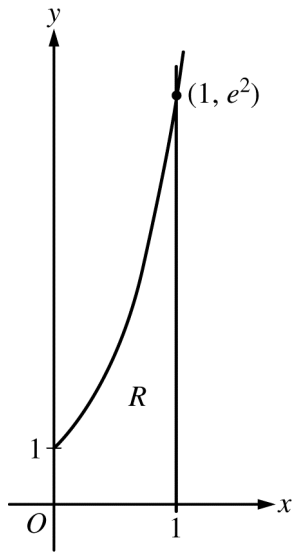

3. Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of $y = f(x)$ and the vertical line $x = 1$, as shown in the figure above.

   (a) Write an equation for the line tangent to the graph of f at $x = 1$.

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Continue problem 3 on page 15.

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(b) Find the area of R.

(c) Region R forms the base of a solid whose cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

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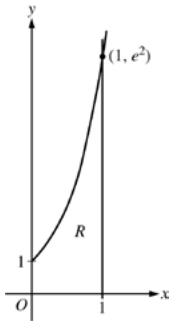
Question 3

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of $y = f(x)$ and the vertical line $x = 1$, as shown in the figure above.



(a) Write an equation for the line tangent to the graph of f at $x = 1$.

(b) Find the area of R.

(c) Region R forms the base of a solid whose cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(a) $f(1) = e^2$

$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$

An equation for the tangent line is $y = e^2 + 2e^2(x - 1)$.

$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$

(b) $\text{Area} = \int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_{x=0}^{x=1} = \frac{1}{2}\left(e^2 - 1\right)$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\text{Volume} = 1 + \int_1^{e^2} \left(1 - \frac{1}{2}\ln y\right)^2 dy$

$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

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t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

   (a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

   (b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

   (c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

   (d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

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Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min^2

1 : average acceleration

(b) v_A is differentiable $\Rightarrow$ v_A is continuous
$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

2 : { 1 : $v_A(8) < -100 < v_A(5)$; 1 : conclusion, using IVT }

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \dfrac{100 + 40}{2} + 3 \cdot \dfrac{40 - 120}{2} + 4 \cdot \dfrac{-120 - 150}{2}$
$= -450$

$s_A(12) \approx 300 - 450 = -150$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

3 : { 1 : position expression; 1 : trapezoidal sum; 1 : position at time $t = 12$ }

(d) Let x be train A's position, y train B's position, and z the distance between train A and train B.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300,\ y = 400 \Rightarrow z = 500$

$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

3 : { 2 : implicit differentiation of distance relationship; 1 : answer }

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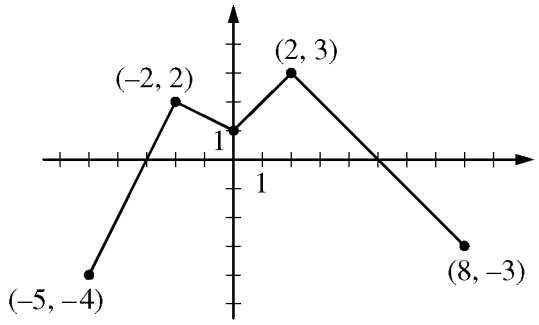

Graph of f

4. The continuous function f is defined on the interval $-5 \le x \le 8$. The graph of f, which consists of four line segments, is shown in the figure above. Let g be the function given by $g(x) = 2x + \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g(-5)$.

(b) Find $g'(x)$ in terms of $f(x)$. For each of $g''(4)$ and $g''(-2)$, find the value or state that it does not exist.

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Continue problem 4 on page 17.

(c) On what intervals, if any, is the graph of g concave down? Give a reason for your answer.

(d) The function h is given by $h(x) = g\left(x^3 + 1\right)$. Find $h'(1)$. Show the work that leads to your answer.

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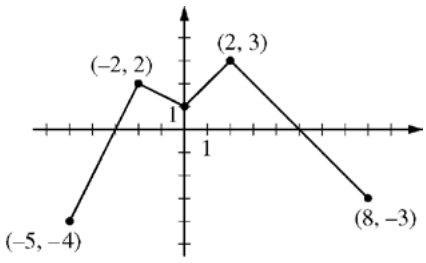
Question 4

The continuous function f is defined on the interval $-5 \le x \le 8$. The graph of f, which consists of four line segments, is shown in the figure above.

Let g be the function given by $g(x) = 2x + \int_{-2}^{x} f(t)\,dt$.



Graph of f

(a) Find $g(0)$ and $g(-5)$.

(b) Find $g'(x)$ in terms of $f(x)$. For each of $g''(4)$ and $g''(-2)$, find the value or state that it does not exist.

(c) On what intervals, if any, is the graph of g concave down? Give a reason for your answer.

(d) The function h is given by $h(x) = g\left(x^3 + 1\right)$. Find $h'(1)$. Show the work that leads to your answer.

(a) $g(0) = 2 \cdot 0 + \int_{-2}^{0} f(t)\,dt = 3$

$g(-5) = 2 \cdot (-5) + \int_{-2}^{-5} f(t)\,dt = -10 + 3 = -7$

$2 : \begin{cases} 1 : g(0) \\ 1 : g(-5) \end{cases}$

(b) $g'(x) = 2 + f(x)$
$g''(x) = f'(x)$

$g''(4) = f'(4) = -1$

$g''(-2) = f'(-2)$ does not exist.

$3 : \begin{cases} 1 : g'(x) \\ 1 : g''(4) \\ 1 : g''(-2) \text{ does not exist} \end{cases}$

(c) The graph of g is concave down on the intervals $(-2, 0)$ and $(2, 8)$ since $g'(x) = 2 + f(x)$ decreases on those intervals.

1 : intervals and reason

(d) $h'(x) = g'\left(x^3 + 1\right) \cdot 3x^2$

$h'(1) = g'(2) \cdot 3 = (2 + f(2)) \cdot 3$
$= (2 + 3) \cdot 3 = 15$

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

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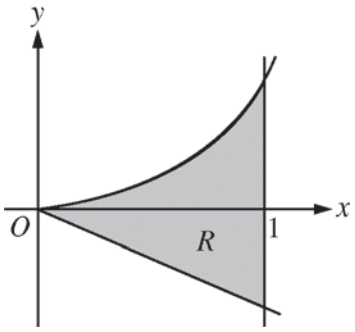

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

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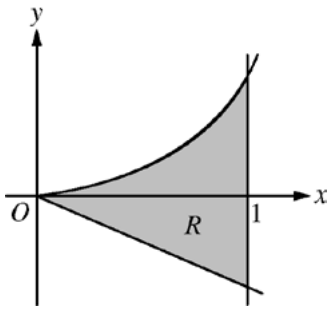
Question 5

Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a) $\text{Area} = \displaystyle\int_0^1 \left(xe^{x^2} - (-2x) \right) dx$

$= \left[\frac{1}{2}e^{x^2} + x^2 \right]_{x=0}^{x=1}$

$= \left(\frac{1}{2}e + 1 \right) - \frac{1}{2} = \frac{e+1}{2}$

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(b) $\text{Volume} = \pi \displaystyle\int_0^1 \left[\left(xe^{x^2} + 2 \right)^2 - (-2x + 2)^2 \right] dx$

3 : { 2 : integrand; 1 : limits and constant }

(c) $y' = \frac{d}{dx}\left(xe^{x^2} \right) = e^{x^2} + 2x^2e^{x^2} = e^{x^2}\left(1 + 2x^2 \right)$

$\text{Perimeter} = \sqrt{5} + 2 + e + \displaystyle\int_0^1 \sqrt{1 + \left[e^{x^2}\left(1 + 2x^2 \right) \right]^2}\, dx$

3 : { 1 : $y' = e^{x^2}\left(1 + 2x^2 \right)$; 1 : integral; 1 : answer }

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5. A toy train moves along a straight track set up on a table. The position $x(t)$ of the train at time t seconds is measured in centimeters from the center of the track. At time $t = 1$, the train is 6 centimeters to the left of the center, so $x(1) = -6$. For $0 \le t \le 4$, the velocity of the train at time t is given by $v(t) = 3t^2 - 12$, where $v(t)$ is measured in centimeters per second.

   (a) For $0 \le t \le 4$, find $x(t)$.

   (b) Find the total distance traveled by the train during the time interval $0 \le t \le 4$.

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Continue problem 5 on page 19.

(c) A toy bus moving on the same table has position given by $(x(t), y(t))$. Here, $x(t)$ is the function found in part (a), and $y(t) = 2 + 12\sin\left(\frac{t}{4}\right)$ is the distance from the bus to the train track, in centimeters. Write, but do not evaluate, an integral expression that gives the total distance traveled by the bus during the time interval $0 \le t \le 4$.

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Question 5

A toy train moves along a straight track set up on a table. The position $x(t)$ of the train at time t seconds is measured in centimeters from the center of the track. At time $t = 1$, the train is 6 centimeters to the left of the center, so $x(1) = -6$. For $0 \le t \le 4$, the velocity of the train at time t is given by $v(t) = 3t^2 - 12$, where $v(t)$ is measured in centimeters per second.

(a) For $0 \le t \le 4$, find $x(t)$.

(b) Find the total distance traveled by the train during the time interval $0 \le t \le 4$.

(c) A toy bus moving on the same table has position given by $(x(t), y(t))$. Here, $x(t)$ is the function found in part (a), and $y(t) = 2 + 12\sin\left(\frac{t}{4}\right)$ is the distance from the bus to the train track, in centimeters. Write, but do not evaluate, an integral expression that gives the total distance traveled by the bus during the time interval $0 \le t \le 4$.

(a) $x(t) = -6 + \int_1^t \left(3u^2 - 12\right) du$

$= -6 + \left[u^3 - 12u\right]_{u=1}^{u=t}$

$= t^3 - 12t + 5$

3 : { 1 : integral; 1 : antiderivative; 1 : answer }

(b) $v(t) = 3\left(t^2 - 4\right) = 0 \Rightarrow t = -2,\ t = 2$

Distance $= |x(2) - x(0)| + |x(4) - x(2)|$

$= |-11 - 5| + |21 - (-11)|$

$= 16 + 32 = 48$

3 : { 1 : identifies $t = 2$; 1 : considers $x(0)$, $x(2)$, and $x(4)$; 1 : answer }

(c) $y'(t) = 3\cos\left(\frac{t}{4}\right)$

Distance $= \int_0^4 \sqrt{\left(3t^2 - 12\right)^2 + \left(3\cos\left(\frac{t}{4}\right)\right)^2}\, dt$

3 : { 1 : $y'(t)$; 2 : integral }

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6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

STOP

END OF EXAM

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Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\,2^{n+1}\,(x-1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}\,2^n\,(x-1)^n}$$
$$= \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-2n(x-1)}{n+1}\right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

3 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : determines radius of convergence }

(b) The first three nonzero terms are $2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1}\,2^n\,(x-1)^{n-1}$ for $n \geq 1$.

3 : { 2 : first three nonzero terms; 1 : general term }

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$

$\ln|1| + C = 0 \Rightarrow C = 0$

$f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$

3 : { 1 : $f'(x)$; 1 : antiderivative; 1 : $f(x)$ }

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6. Let $a_n = \dfrac{1}{n \ln n}$ for $n \geq 3$.

(a) Let f be the function given by $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 3$, f is continuous, decreasing, and positive. Use the integral test to show that $\displaystyle\sum_{n=3}^{\infty} a_n$ diverges.

(b) Consider the infinite series $\displaystyle\sum_{n=3}^{\infty} (-1)^{n+1} a_n = \frac{1}{3 \ln 3} - \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} - \cdots$. Identify properties of this series that guarantee the series converges. Explain why the sum of this series is less than $\dfrac{1}{3}$.

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Continue problem 6 on page 21.

(c) Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$. Show the analysis that leads to your answer.

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Question 6

Let $a_n = \dfrac{1}{n \ln n}$ for $n \geq 3$.

(a) Let f be the function given by $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 3$, f is continuous, decreasing, and positive.

Use the integral test to show that $\displaystyle\sum_{n=3}^{\infty} a_n$ diverges.

(b) Consider the infinite series $\displaystyle\sum_{n=3}^{\infty} (-1)^{n+1} a_n = \frac{1}{3\ln 3} - \frac{1}{4\ln 4} + \frac{1}{5\ln 5} - \cdots$. Identify properties of this series that guarantee the series converges. Explain why the sum of this series is less than $\dfrac{1}{3}$.

(c) Find the interval of convergence of the power series $\displaystyle\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$. Show the analysis that leads to your answer.

(a) $u = \ln x \Rightarrow du = \dfrac{1}{x}\,dx$

$$\int_3^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{1}{u}\,du = \lim_{b\to\infty} \Big[\ln|u|\Big]_{u=\ln 3}^{u=\ln b}$$
$$= \lim_{b\to\infty} \left(\ln|\ln b| - \ln|\ln 3|\right) = \infty$$

Therefore, the series diverges.

3 : { 1 : integral; 1 : antiderivative; 1 : evaluation }

(b) The terms in this alternating series decrease in absolute value and $\displaystyle\lim_{n\to\infty} \frac{1}{n \ln n} = 0$. Therefore, the Alternating Series Test guarantees that this series converges. Furthermore,

$$\frac{1}{3\ln 3} - \frac{1}{4\ln 4} < \text{Sum} < \frac{1}{3\ln 3} < \frac{1}{3}$$

Therefore, the sum of the series is less than $\dfrac{1}{3}$.

2 : { 1 : properties; 1 : explanation }

(c) $$\left|\frac{(x-2)^{n+2}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(x-2)^{n+1}}\right| = \left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)|x-2|$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)|x-2| = |x-2|$$

$|x-2| < 1 \Rightarrow 1 < x < 3$

When $x = 1$, the series is $\dfrac{1}{3\ln 3} - \dfrac{1}{4\ln 4} + \dfrac{1}{5\ln 5} - \dfrac{1}{6\ln 6} + \cdots$

This series converges by the Alternating Series Test.

When $x = 3$, the series is $\dfrac{1}{3\ln 3} + \dfrac{1}{4\ln 4} + \dfrac{1}{5\ln 5} + \dfrac{1}{6\ln 6} + \cdots$

This series diverges by the integral test, as shown in part (a).

Therefore, the interval of convergence is $1 \leq x < 3$.

4 : { 1: sets up ratio; 1: computes limit of ratio; 1: identifies radius of convergence; 1: analysis and interval of convergence }

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CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

   (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

   (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

   (c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

   (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

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Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

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(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t)\,dt = 76.570$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$
Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

$2 : \begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t, $0 \le t \le 8$, is $30 + \int_0^t [R(x) - D(x)]\,dx$.

$R(t) - D(t) = 0 \Rightarrow t = 0,\ 3.271658$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

$3 : \begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $30 + \int_0^w [R(t) - D(t)]\,dt = 50$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

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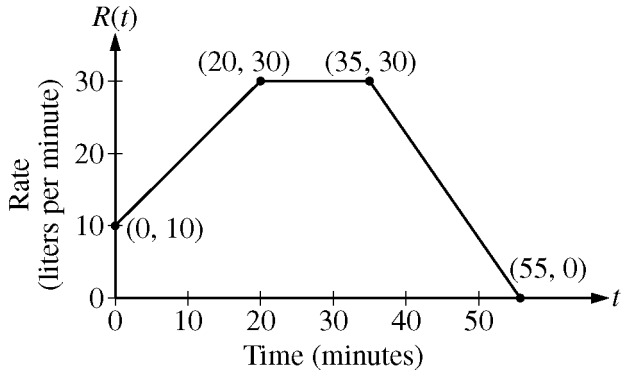
A graphing calculator is required for these problems.



1. At time $t = 0$ minutes, a tank contains 100 liters of water. The piecewise-linear graph above shows the rate $R(t)$, in liters per minute, at which water is pumped into the tank during a 55-minute period.

   (a) Find $R'(45)$. Using appropriate units, explain the meaning of your answer in the context of this problem.

   (b) How many liters of water have been pumped into the tank from time $t = 0$ to time $t = 55$ minutes? Show the work that leads to your answer.

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Continue problem 1 on page 5.

A graphing calculator is required for these problems.

(c) At time $t = 10$ minutes, water begins draining from the tank at a rate modeled by the function D, where $D(t) = 10e^{(\sin t)/10}$ liters per minute. Water continues to drain at this rate until time $t = 55$ minutes. How many liters of water are in the tank at time $t = 55$ minutes?

(d) Using the functions R and D, determine whether the amount of water in the tank is increasing or decreasing at time $t = 45$ minutes. Justify your answer.

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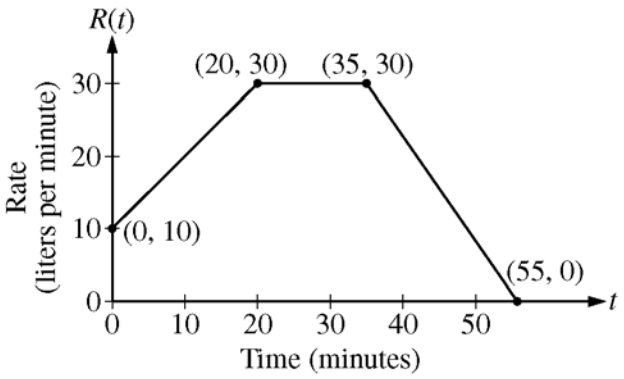
Question 1

At time $t = 0$ minutes, a tank contains 100 liters of water. The piecewise-linear graph above shows the rate $R(t)$, in liters per minute, at which water is pumped into the tank during a 55-minute period.



(a) Find $R'(45)$. Using appropriate units, explain the meaning of your answer in the context of this problem.

(b) How many liters of water have been pumped into the tank from time $t = 0$ to time $t = 55$ minutes? Show the work that leads to your answer.

(c) At time $t = 10$ minutes, water begins draining from the tank at a rate modeled by the function D, where $D(t) = 10e^{(\sin t)/10}$ liters per minute. Water continues to drain at this rate until time $t = 55$ minutes. How many liters of water are in the tank at time $t = 55$ minutes?

(d) Using the functions R and D, determine whether the amount of water in the tank is increasing or decreasing at time $t = 45$ minutes. Justify your answer.

(a) $R'(45) = \dfrac{30 - 0}{35 - 55} = -\dfrac{3}{2}$

The rate at which water is being pumped into the tank is decreasing at $\dfrac{3}{2}$ liters/min^2 at $t = 45$ minutes.

$2 : \begin{cases} 1 : R'(45) \\ 1 : \text{explanation} \end{cases}$

(b) $\displaystyle\int_0^{55} R(t)\,dt = 20 \cdot \frac{10 + 30}{2} + 15 \cdot 30 + \frac{1}{2} \cdot 20 \cdot 30$

$= 400 + 450 + 300 = 1150$

$2 : \begin{cases} 1 : \text{sum of areas} \\ 1 : \text{answer} \end{cases}$

(c) $\text{Amt} = 100 + 1150 - \displaystyle\int_{10}^{55} 10e^{(\sin t)/10}\,dt$

$= 1250 - 450.275371 = 799.725$ (or 799.724)

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{expression for water in the tank} \\ 1 : \text{answer} \end{cases}$

(d) $R(45) = 15$

$D(45) = 10.88815$

At time $t = 45$ minutes, the rate of water pumped into the tank is greater than the rate of water draining from the tank. Therefore, the amount of water in the tank is increasing at time $t = 45$ minutes.

$2 :$ answer with justification

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2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

   (a) Find the x-coordinate of the position of the particle at time $t = 2$.

   (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

   (c) Find the time at which the speed of the particle is 3.

   (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

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Question 2

At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

(a) $x(2) = 3 + \int_1^2 \cos\left(t^2\right)\, dt = 2.557$ (or 2.556)

3 : { 1 : integral; 1 : uses initial condition; 1 : answer }

(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^{0.5t}}{\cos\left(t^2\right)}$

$\dfrac{e^{0.5t}}{\cos\left(t^2\right)} = 2$

$t = 0.840$

2 : { 1 : slope in terms of t; 1 : answer }

(c) Speed $= \sqrt{\cos^2\left(t^2\right) + e^t}$

$\sqrt{\cos^2\left(t^2\right) + e^t} = 3$

$t = 2.196$ (or 2.195)

2 : { 1 : speed in terms of t; 1 : answer }

(d) Distance $= \int_0^1 \sqrt{\cos^2\left(t^2\right) + e^t}\, dt = 1.595$ (or 1.594)

2 : { 1 : integral; 1 : answer }

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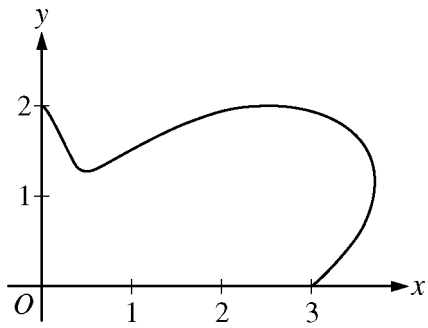

2. The figure above shows the graph of the polar equation $r = 2 + \sin(4\theta) + \cos(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. The derivative of r with respect to θ is given by $r'(\theta) = 4\cos(4\theta) - \sin(\theta)$.

(a) Find the area of the region bounded by the graph of r and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(b) Find the area of the region in the first quadrant that is outside the graph of $r = 2 + \sin(4\theta) + \cos(\theta)$ but inside the graph of the circle of radius 2 centered at the origin.

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Continue problem 2 on page 7.

(c) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve $r = 2 + \sin(4\theta) + \cos(\theta)$ with greatest distance from the origin. Justify your answer.

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CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

   (a) Use the data in the table to estimate the value of $v'(16)$.

   (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

   Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

   (c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

   Find Bob's acceleration at time $t = 5$.

   (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

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Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

(a) $v'(16) \approx \dfrac{240 - 200}{20 - 12} = 5$ meters/min^2

1 : approximation

(b) $\int_0^{40} |v(t)|\,dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \le t \le 40$ minutes.

$$\begin{aligned}\int_0^{40} |v(t)|\,dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters}\end{aligned}$$

3 : { 1 : explanation; 1 : right Riemann sum; 1 : approximation }

(c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$B'(5) = 3(25) - 12(5) = 15$ meters/min^2

2 : { 1 : uses $B'(t)$; 1 : answer }

(d) $\text{Avg vel} = \dfrac{1}{10}\displaystyle\int_0^{10} \left(t^3 - 6t^2 + 300\right) dt$

$$= \frac{1}{10}\left[\frac{t^4}{4} - 2t^3 + 300t\right]_0^{10}$$

$$= \frac{1}{10}\left[\frac{10000}{4} - 2000 + 3000\right] = 350 \text{ meters/min}$$

3 : { 1 : integral; 1 : antiderivative; 1 : answer }

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t (seconds)	0	3	5	8	12
$k(t)$ (feet per second)	0	5	10	20	24

3. Kathleen skates on a straight track. She starts from rest at the starting line at time $t = 0$. For $0 < t \le 12$ seconds, Kathleen's velocity k, measured in feet per second, is differentiable and increasing. Values of $k(t)$ at various times t are given in the table above.

   (a) Use the data in the table to estimate Kathleen's acceleration at time $t = 4$ seconds. Show the computations that lead to your answer. Indicate units of measure.

   (b) Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_0^{12} k(t)\,dt$. Indicate units of measure. Is this approximation an overestimate or an underestimate for the value of $\int_0^{12} k(t)\,dt$? Explain your reasoning.

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Continue problem 3 on page 15.

(c) Nathan skates on the same track, starting 5 feet ahead of Kathleen at time $t = 0$. Nathan's velocity, in feet per second, is given by $n(t) = \frac{150}{t+3} - 50e^{-t}$. Write, but do not evaluate, an expression involving an integral that gives Nathan's distance from the starting line at time $t = 12$ seconds.

(d) Write an expression for Nathan's acceleration in terms of t.

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Question 3

t (seconds)	0	3	5	8	12
$k(t)$ (feet per second)	0	5	10	20	24

Kathleen skates on a straight track. She starts from rest at the starting line at time $t = 0$. For $0 < t \le 12$ seconds, Kathleen's velocity k, measured in feet per second, is differentiable and increasing. Values of $k(t)$ at various times t are given in the table above.

(a) Use the data in the table to estimate Kathleen's acceleration at time $t = 4$ seconds. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_0^{12} k(t)\,dt$. Indicate units of measure. Is this approximation an overestimate or an underestimate for the value of $\int_0^{12} k(t)\,dt$? Explain your reasoning.

(c) Nathan skates on the same track, starting 5 feet ahead of Kathleen at time $t = 0$. Nathan's velocity, in feet per second, is given by $n(t) = \frac{150}{t+3} - 50e^{-t}$. Write, but do not evaluate, an expression involving an integral that gives Nathan's distance from the starting line at time $t = 12$ seconds.

(d) Write an expression for Nathan's acceleration in terms of t.

(a) $a(4) \approx \frac{10-5}{5-3} = \frac{5}{2}$ ft/sec^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) $\int_0^{12} k(t)\,dt \approx (5)(3) + (10)(2) + (20)(3) + (24)(4) = 191$ feet

This approximation is an overestimate since a right Riemann sum is used and the function k is increasing.

$3 : \begin{cases} 1 : \text{right Riemann sum} \\ 1 : \text{approximation with units} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $s(12) = 5 + \int_0^{12} n(t)\,dt$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $n'(t) = (150)(-1)(t+3)^{-2} - 50e^{-t}(-1)$

$= -\frac{150}{(t+3)^2} + 50e^{-t}$

$2 : n'(t)$

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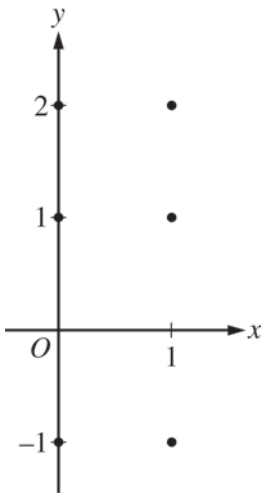
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

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Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

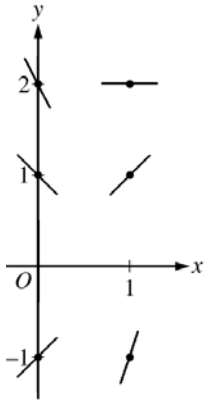
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } \left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

$b = -m \Rightarrow b = -2$

Therefore, $m = 2$ and $b = -2$.

$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

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4. In a national park, the population of mountain lions grows over time. At time $t = 0$, where t is measured in years, the population is found to be 20 mountain lions.

   (a) One zoologist suggests a population model P that satisfies the differential equation $\frac{dP}{dt} = \frac{1}{4}(220 - P)$.

   Use separation of variables to solve this differential equation for P with the initial condition $P(0) = 20$.

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Continue problem 4 on page 17.

(b) A second zoologist suggests a population model Q that satisfies $\frac{dQ}{dt} = \frac{1}{500}Q(220 - Q)$. Find the value of $\frac{dQ}{dt}$ at the time when Q grows most rapidly.

(c) For the population model Q introduced in part (b), use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $Q(10)$. Show the computations that lead to your answer.

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Question 4

In a national park, the population of mountain lions grows over time. At time $t = 0$, where t is measured in years, the population is found to be 20 mountain lions.

(a) One zoologist suggests a population model P that satisfies the differential equation $\frac{dP}{dt} = \frac{1}{4}(220 - P)$. Use separation of variables to solve this differential equation for P with the initial condition $P(0) = 20$.

(b) A second zoologist suggests a population model Q that satisfies $\frac{dQ}{dt} = \frac{1}{500}Q(220 - Q)$. Find the value of $\frac{dQ}{dt}$ at the time when Q grows most rapidly.

(c) For the population model Q introduced in part (b), use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $Q(10)$. Show the computations that lead to your answer.

(a) $\frac{dP}{dt} = \frac{1}{4}(220 - P)$

$\int \frac{dP}{220 - P} = \int \frac{1}{4}\, dt$

$-\ln|220 - P| = \frac{1}{4}t + C$

Because $P(0) = 20$, $P < 220$, so $|220 - P| = 220 - P$.

$-\ln(220 - 20) = \frac{1}{4}(0) + C \Rightarrow C = -\ln 200$

$220 - P = 200e^{-t/4}$

$P = 220 - 200e^{-t/4}$, $t \geq 0$

5 : { 1 : separation of variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for P }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) Q satisfies a logistic differential equation with carrying capacity 220. Q grows most rapidly when $Q = \frac{220}{2} = 110$.

$\left.\frac{dQ}{dt}\right|_{Q=110} = \frac{110^2}{500} = \frac{121}{5}$

2 : { 1 : $Q = 110$; 1 : answer }

(c) $Q(0) = 20$

$Q'(0) = \frac{1}{500}(20)(200) = 8$

$Q(5) \approx 20 + (8)(5) = 60$

$Q'(5) \approx \frac{1}{500}(60)(220 - 60) = \frac{96}{5}$

$Q(10) \approx 60 + \left(\frac{96}{5}\right)(5) = 156$

2 : { 1 : Euler's method with two steps; 1 : answer }

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5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

   (a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

   (b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

   (c) Find the value of k for which f has a critical point at $x = -5$.

   (d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.
   Find $\int f(x)\,dx$.

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Question 5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

(a) $f(4) = \dfrac{1}{4^2 - 3 \cdot 4} = \dfrac{1}{4}$ $\qquad f'(4) = \dfrac{3 - 2 \cdot 4}{\left(4^2 - 3 \cdot 4\right)^2} = -\dfrac{5}{16}$

An equation for the line tangent to the graph of f at the point whose x-coordinate is 4 is $y = -\dfrac{5}{16}(x - 4) + \dfrac{1}{4}$.

2 : { 1 : slope; 1 : tangent line equation }

(b) $f'(x) = \dfrac{4 - 2x}{\left(x^2 - 4x\right)^2}$ $\qquad f'(2) = \dfrac{4 - 2 \cdot 2}{\left(2^2 - 4 \cdot 2\right)^2} = 0$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

2 : { 1 : considers $f'(2)$; 1 : answer with justification }

(c) $f'(-5) = \dfrac{k - 2 \cdot (-5)}{\left((-5)^2 - k \cdot (-5)\right)^2} = 0 \Rightarrow k = -10$

1 : answer

(d) $\dfrac{1}{x^2 - 6x} = \dfrac{1}{x(x - 6)} = \dfrac{A}{x} + \dfrac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\dfrac{1}{6}$

$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \dfrac{1}{6}$

$\dfrac{1}{x(x - 6)} = \dfrac{-1/6}{x} + \dfrac{1/6}{x - 6}$

$\int f(x)\,dx = \int \left(\dfrac{-1/6}{x} + \dfrac{1/6}{x - 6} \right) dx$

$= -\dfrac{1}{6}\ln|x| + \dfrac{1}{6}\ln|x - 6| + C = \dfrac{1}{6}\ln\left|\dfrac{x - 6}{x}\right| + C$

4 : { 2 : partial fraction decomposition; 2 : general antiderivative }

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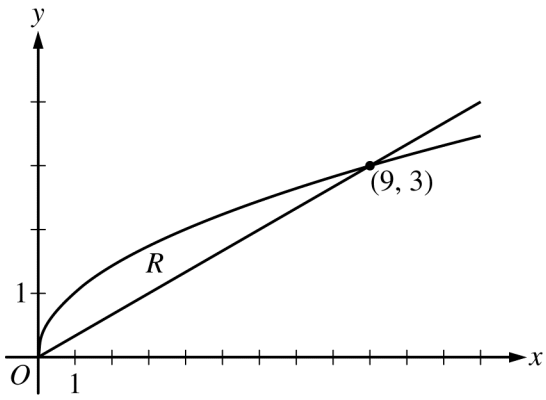

5. Let R be the region in the first quadrant enclosed by the graphs of $g(x) = \sqrt{x}$ and $h(x) = \frac{x}{3}$, as shown in the figure above.

   (a) Find the area of region R.

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Continue problem 5 on page 19.

(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when R is revolved about the horizontal line $y = 4$.

(c) Find the maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$. Justify your answer.

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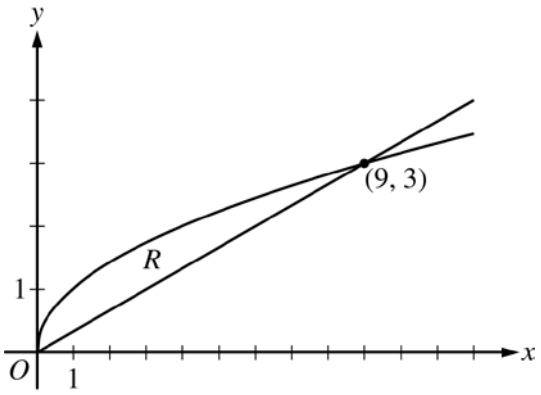
Question 5

Let R be the region in the first quadrant enclosed by the graphs of $g(x) = \sqrt{x}$ and $h(x) = \frac{x}{3}$, as shown in the figure above.

(a) Find the area of region R.

(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when R is revolved about the horizontal line $y = 4$.

(c) Find the maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$. Justify your answer.



(a) Area $= \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right]_0^9$

$= \frac{2}{3} \cdot 27 - \frac{1}{6} \cdot 81 = \frac{9}{2}$

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(b) Volume $= \pi \int_0^9 \left[\left(4 - \frac{x}{3}\right)^2 - \left(4 - \sqrt{x}\right)^2\right] dx$

3 : { 2 : integrand; 1 : limits and constant }

(c) Consider the function $D(x) = \sqrt{x} - \frac{x}{3}$.

$D'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3} = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$D'(x) = 0 \Rightarrow x = \frac{9}{4}$

x	$D(x)$	Distance between graphs
0	0	0
$\frac{9}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
16	$-\frac{4}{3}$	$\frac{4}{3}$

The maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$ is $\frac{4}{3}$.

3 : { 1 : sets $D'(x) = 0$; 1 : identifies $x = \frac{9}{4}$ as a candidate; 1 : answer and justification }

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6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

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Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

(a) Let a_n be the nth term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n\to\infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

3 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : determines radius of convergence }

(b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$$

3 : { 2 : first four nonzero terms; 1 : rational function }

(c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right)$$

$$= x - \frac{1}{2}x^2 + 2x^3 + \cdots$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

3 : { 1 : first four nonzero terms of the Maclaurin series for e^x; 2 : Taylor polynomial }

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6. The Maclaurin series for a function f is given by $\frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \cdots + \frac{x^n}{n+2} + \cdots$.

   (a) Use the ratio test to find the interval of convergence of the Maclaurin series for f.

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Continue problem 6 on page 21.

(b) Let g be the function given by $g(x) = f(-2x)$. Find the first three terms and the general term of the Maclaurin series for g.

(c) The first two terms of the Maclaurin series for f are used to approximate $f(0.1)$. Given that $|f'''(x)| \le 2$ for $0 \le x \le 0.1$, use the Lagrange error bound to show that this approximation differs from $f(0.1)$ by at most $\frac{1}{3000}$.

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Question 6

The Maclaurin series for a function f is given by $\frac{x}{3}+\frac{x^2}{4}+\frac{x^3}{5}+\cdots+\frac{x^n}{n+2}+\cdots$.

(a) Use the ratio test to find the interval of convergence of the Maclaurin series for f.

(b) Let g be the function given by $g(x)=f(-2x)$. Find the first three terms and the general term of the Maclaurin series for g.

(c) The first two terms of the Maclaurin series for f are used to approximate $f(0.1)$. Given that $|f'''(x)|\le 2$ for $0\le x\le 0.1$, use the Lagrange error bound to show that this approximation differs from $f(0.1)$ by at most $\frac{1}{3000}$.

(a) Let a_n be the nth term of the Maclaurin series.

$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{n+3}\cdot\frac{n+2}{x^n}=\left(\frac{n+2}{n+3}\right)\cdot x$

$\lim_{n\to\infty}\left|\left(\frac{n+2}{n+3}\right)\cdot x\right|=|x|$

$|x|<1 \Rightarrow -1<x<1$

The series converges when $-1<x<1$.

When $x=-1$, the series is $-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\cdots$.

This series converges by the alternating series test.

When $x=1$, the series is $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$.

This series diverges since it is a harmonic series.

Therefore, the interval of convergence is $-1\le x<1$.

5 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : identifies interior of interval of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

(b) The first three terms are

$\frac{-2x}{3}+\frac{(-2x)^2}{4}+\frac{(-2x)^3}{5}=-\frac{2}{3}x+x^2-\frac{8}{5}x^3$.

The general term is $\frac{(-2x)^n}{n+2}$.

2 : { 1 : first three terms; 1 : general term }

(c) Error $\le \frac{\max\limits_{0\le x\le 0.1}|f'''(x)|}{3!}\cdot\left(\frac{1}{10}\right)^3\le\frac{2}{6}\cdot\frac{1}{1000}=\frac{1}{3000}$

2 : { 1 : form of the error bound; 1 : analysis }

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CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

   (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

   (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

   (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

   (d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

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Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned} \int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $\text{Total} \approx 50000 + \int_0^8 W(t)\,dt - 8050$

$= 50000 + 7836.195325 - 8050 \approx 49786$ liters

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Therefore, the Intermediate Value Theorem guarantees at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

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1. The curve C in the xy-plane is given parametrically by $(x(t), y(t))$, where $\frac{dx}{dt} = t \sin\left(\frac{2t}{3}\right)$ and $\frac{dy}{dt} = \cos\left(\frac{t^2}{4}\right)$ for $1 \le t \le 6$.

   (a) Find the slope of the line tangent to the curve C at the point where $t = 3$.

   (b) For $1 \le t \le 6$, what is the value of t at which the line tangent to the curve C is vertical?

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Continue problem 1 on page 5.

(c) Find the length of the curve C for $1 \leq t \leq 6$.

(d) Given $y(1) = 2$, find $y(3)$.

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Question 1

The curve C in the xy-plane is given parametrically by $(x(t), y(t))$, where $\frac{dx}{dt} = t\sin\left(\frac{2t}{3}\right)$ and $\frac{dy}{dt} = \cos\left(\frac{t^2}{4}\right)$ for $1 \le t \le 6$.

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(b) For $1 \le t \le 6$, what is the value of t at which the line tangent to the curve C is vertical?

(c) Find the length of the curve C for $1 \le t \le 6$.

(d) Given $y(1) = 2$, find $y(3)$.

(a) Slope $= \frac{y'(3)}{x'(3)} = \frac{\cos(9/4)}{3\sin(2)} = -0.230$

$2 : \begin{cases} 1 : \text{expression for the slope} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{dx}{dt} = t\sin\left(\frac{2t}{3}\right) = 0 \Rightarrow t = \frac{3\pi}{2} \approx 4.712$

$2 : \begin{cases} 1 : \text{sets } \frac{dx}{dt} = 0 \\ 1 : \text{answer} \end{cases}$

(c) Length $= \int_1^6 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 10.569$ (or 10.568)

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $y(3) = 2 + \int_1^3 y'(t)\, dt = 2.805$ (or 2.804)

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{initial condition} \\ 1 : \text{answer} \end{cases}$

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t (minutes)	0	2	5	7	10
$h(t)$ (inches)	3.5	10.0	15.5	18.5	20.0

1. The depth of water in tank A, in inches, is modeled by a differentiable and increasing function h for $0 \le t \le 10$, where t is measured in minutes. Values of $h(t)$ for selected values of t are given in the table above.

   (a) Use the data in the table to find an approximation for $h'(6)$. Show the computations that lead to your answer. Indicate units of measure.

   (b) Approximate the value of $\int_0^{10} h(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the data in the table. Is this approximation greater than or less than $\int_0^{10} h(t)\,dt$? Give a reason for your answer.

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(c) The depth of water in tank B, in inches, is modeled by the function $g(t) = 3.2 + 17.5\sqrt{\sin(0.16t)}$ for $0 \le t \le 10$, where t is measured in minutes. Find the average depth of the water in tank B over the interval $0 \le t \le 10$. Is this value greater than or less than the average depth of the water in tank A over the interval $0 \le t \le 10$? Give a reason for your answer.

(d) According to the model given in part (c), is the depth of the water in tank B increasing or decreasing at time $t = 6$? Give a reason for your answer.

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GO ON TO THE NEXT PAGE.

Question 1

(a) $h'(6) \approx \dfrac{h(7) - h(5)}{7 - 5} = \dfrac{18.5 - 15.5}{2} = 1.5$ in/min

1 : answer with units

(b) $\displaystyle\int_0^{10} h(t)\,dt \approx (2 - 0) \cdot h(2) + (5 - 2) \cdot h(5) + (7 - 5) \cdot h(7) + (10 - 7) \cdot h(10)$
$= 2(10.0) + 3(15.5) + 2(18.5) + 3(20.0) = 163.5$

Because h is an increasing function, the right Riemann sum approximation is greater than $\displaystyle\int_0^{10} h(t)\,dt$.

3 : { 1 : right Riemann sum; 1 : approximation; 1 : overestimate with reason }

(c) Average depth in tank $B = \dfrac{1}{10}\displaystyle\int_0^{10} g(t)\,dt = 16.624$ in

Average depth in tank $A = \dfrac{1}{10}\displaystyle\int_0^{10} h(t)\,dt < \dfrac{1}{10}(163.5) = 16.35$ in < 16.624 in

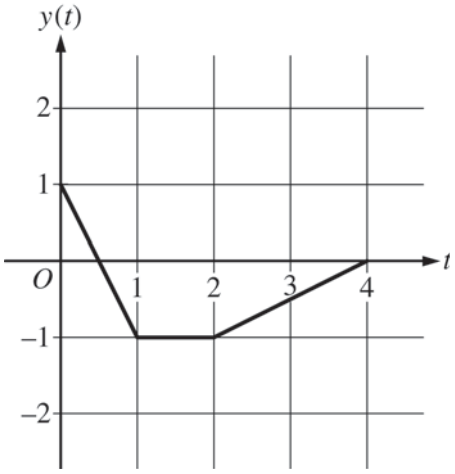
Therefore, the average depth of the water in tank B is greater than the average depth of the water in tank A.

3 : { 1 : integral; 1 : average depth in tank B; 1 : answer with reason }

(d) $g'(6) = 0.887$

The depth of the water in tank B is increasing at time $t = 6$ because $g'(6) > 0$.

2 : { 1 : uses $g'(6)$; 1 : answer with reason }



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin\left(3t^2\right)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

   (a) Find the position of the particle at $t = 3$.

   (b) Find the slope of the line tangent to the path of the particle at $t = 3$.

   (c) Find the speed of the particle at $t = 3$.

   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II

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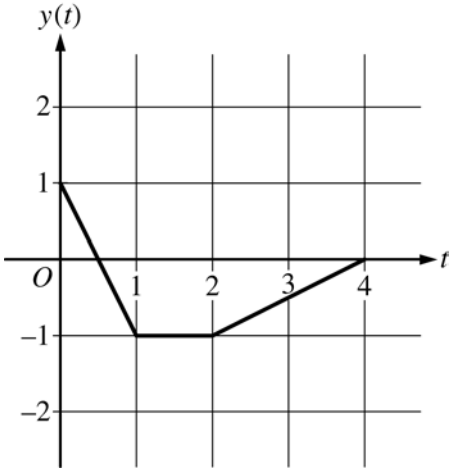
Question 2



At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin\left(3t^2\right)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 = 14.377$

$y(3) = -\frac{1}{2}$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

3 : { 1 : integral; 1 : uses initial condition; 1 : answer }

(b) Slope $= \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

1 : slope

(c) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$ (or 9.968)

2 : { 1 : expression for speed; 1 : answer }

(d) Distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\,dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\,dt$

$= 2.237871 + 2.112003 = 4.350$ (or 4.349)

3 : { 1 : expression for distance; 1 : integrals; 1 : answer }

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A graphing calculator is required for these problems.

t (weeks)	0	3	6	10	12
$G(t)$ (games per week)	160	450	900	2100	2400

2. A store tracks the sales of one of its popular board games over a 12-week period. The rate at which games are being sold is modeled by the differentiable function G, where $G(t)$ is measured in games per week and t is measured in weeks for $0 \le t \le 12$. Values of $G(t)$ are given in the table above for selected values of t.

   (a) Approximate the value of $G'(8)$ using the data in the table. Show the computations that lead to your answer.

   (b) Approximate the value of $\int_0^{12} G(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the table. Explain the meaning of $\int_0^{12} G(t)\,dt$ in the context of this problem.

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Continue problem 2 on page 7.

A graphing calculator is required for these problems.

(c) One salesperson believes that, starting with 2400 games per week at time $t = 12$, the rate at which games will be sold will increase at a constant rate of 100 games per week per week. Based on this model, how many total games will be sold in the 8 weeks between time $t = 12$ and $t = 20$?

(d) Another salesperson believes the best model for the rate at which games will be sold in the 8 weeks between time $t = 12$ and $t = 20$ is $M(t) = 2400e^{-0.01(t-12)^2}$ games per week. Based on this model, how many total games, to the nearest whole number, will be sold during this period?

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Question 2

t (weeks)	0	3	6	10	12
$G(t)$ (games per week)	160	450	900	2100	2400

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(a) Approximate the value of $G'(8)$ using the data in the table. Show the computations that lead to your answer.

(b) Approximate the value of $\int_0^{12} G(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the table. Explain the meaning of $\int_0^{12} G(t)\,dt$ in the context of this problem.

(c) One salesperson believes that, starting with 2400 games per week at time $t = 12$, the rate at which games will be sold will increase at a constant rate of 100 games per week per week. Based on this model, how many total games will be sold in the 8 weeks between time $t = 12$ and $t = 20$?

(d) Another salesperson believes the best model for the rate at which games will be sold in the 8 weeks between time $t = 12$ and $t = 20$ is $M(t) = 2400e^{-0.01(t-12)^2}$ games per week. Based on this model, how many total games, to the nearest whole number, will be sold during this period?

(a) $G'(8) \approx \dfrac{G(10) - G(6)}{10 - 6} = \dfrac{2100 - 900}{10 - 6} = 300$ games per week per week

1 : approximation

(b) $\int_0^{12} G(t)\,dt \approx 3 \cdot G(3) + 3 \cdot G(6) + 4 \cdot G(10) + 2 \cdot G(12)$
$= 3 \cdot 450 + 3 \cdot 900 + 4 \cdot 2100 + 2 \cdot 2400$
$= 17250$

$\int_0^{12} G(t)\,dt$ represents the total number of games sold over the 12-week period $0 \le t \le 12$.

3 : { 1 : right Riemann sum; 1 : approximation; 1 : explanation }

(c) The rate of sales of the game for $12 \le t \le 20$ is $R(t) = 2400 + 100(t - 12)$.

Based on this model, the total number of games that will be sold between time $t = 12$ and $t = 20$ is $\int_{12}^{20} R(t)\,dt = 22400$.

3 : { 1 : rate function; 1 : integral; 1 : answer }

(d) $\int_{12}^{20} M(t)\,dt = 15784.07655$

Based on this model, the total number of games that will be sold between time $t = 12$ and $t = 20$ is 15784.

2 : { 1 : integral; 1 : answer }

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2. For $t \geq 0$, a particle moving in the xy-plane has the position vector $\langle x(t), y(t) \rangle$ at time t, where $\frac{dx}{dt} = -1 + e^{\sin t}$ and $\frac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the position of the particle is $(5, 7)$.

(a) Find the acceleration vector of the particle at time $t = 2$.

(b) Find the total distance traveled by the particle over the time interval $1.8 \leq t \leq 2$.

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(c) Find the x-coordinate of the position of the particle at time $t = 1$.

(d) At time $t = \sqrt{\frac{7\pi}{2}}$, the line tangent to the path of the particle is horizontal. Find the particle's speed at time $t = \sqrt{\frac{7\pi}{2}}$. Determine whether the particle is moving to the left or to the right at that time. Give a reason for your answer.

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Question 2

(a) Acceleration $= \langle x''(2), y''(2) \rangle$
$= \langle -1.033, 3.027 \rangle$

$2 : \begin{cases} 1 : x''(2) \\ 1 : y''(2) \end{cases}$

(b) Distance $= \int_{1.8}^{2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$
$= 0.360$ (or 0.359)

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $x(1) = x(2) + \int_{2}^{1} x'(t)\, dt$
$= 5 + \int_{2}^{1} \left(-1 + e^{\sin t}\right) dt = 3.395$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses } x(2) \\ 1 : \text{answer} \end{cases}$

(d) Speed $= \sqrt{(x'(t))^2 + (y'(t))^2}\Big|_{t=\sqrt{\frac{7\pi}{2}}} = \left|x'\left(\sqrt{\frac{7\pi}{2}}\right)\right| = 0.159$

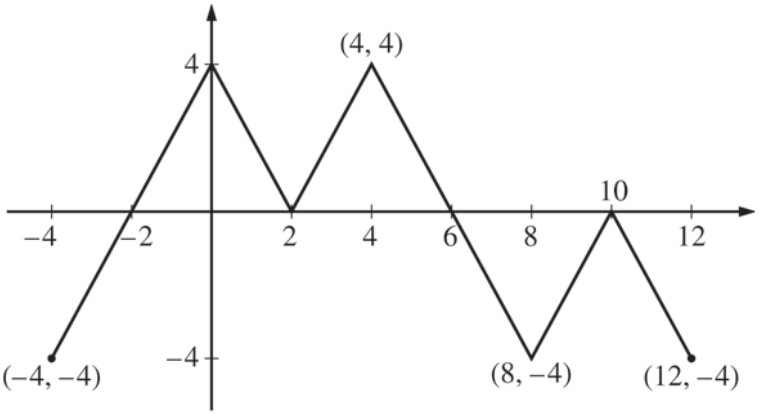
$x'\left(\sqrt{\frac{7\pi}{2}}\right) = -0.159$

Because $x'\left(\sqrt{\frac{7\pi}{2}}\right) < 0$, the particle is moving to the left at time $t = \sqrt{\frac{7\pi}{2}}$.

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{direction with reason} \end{cases}$

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

   (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

   (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

   (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

   (d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

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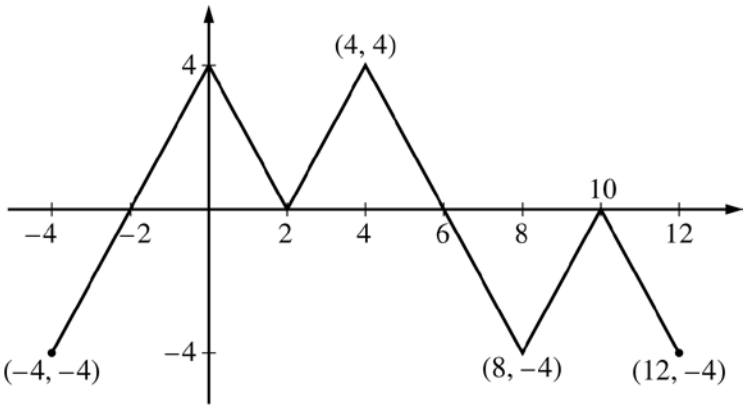
Question 3

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$1 : g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

4 : 1 : considers $x = -2$ and $x = 6$ as candidates
1 : considers $x = -4$ and $x = 12$
2 : answers with justification

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals

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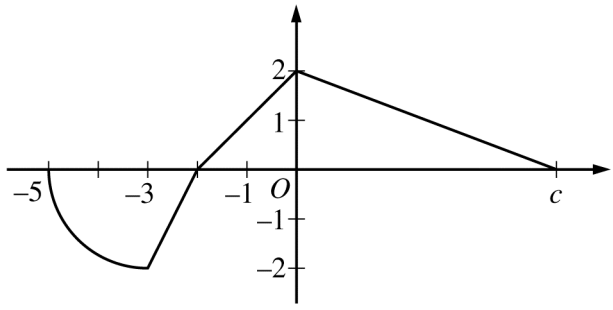

Graph of f

3. The function f is defined on the interval $-5 \le x \le c$, where $c > 0$ and $f(c) = 0$. The graph of f, which consists of three line segments and a quarter of a circle with center $(-3, 0)$ and radius 2, is shown in the figure above.

   (a) Find the average rate of change of f over the interval $[-5, 0]$. Show the computations that lead to your answer.

   (b) For $-5 \le x \le c$, let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the x-coordinate of each point of inflection of the graph of g. Justify your answer.

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Continue problem 3 on page 15.

(c) Find the value of c for which the average value of f over the interval $-5 \le x \le c$ is $\frac{1}{2}$.

(d) Assume $c > 3$. The function h is defined by $h(x) = f\left(\frac{x}{2}\right)$. Find $h'(6)$ in terms of c.

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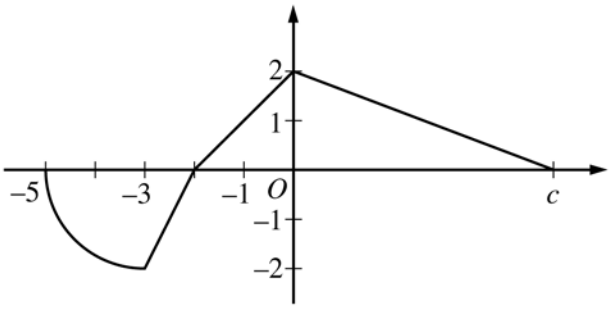
Question 3

The function f is defined on the interval $-5 \le x \le c$, where $c > 0$ and $f(c) = 0$. The graph of f, which consists of three line segments and a quarter of a circle with center $(-3, 0)$ and radius 2, is shown in the figure above.



Graph of f

(a) Find the average rate of change of f over the interval $[-5, 0]$. Show the computations that lead to your answer.

(b) For $-5 \le x \le c$, let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the x-coordinate of each point of inflection of the graph of g. Justify your answer.

(c) Find the value of c for which the average value of f over the interval $-5 \le x \le c$ is $\frac{1}{2}$.

(d) Assume $c > 3$. The function h is defined by $h(x) = f\left(\frac{x}{2}\right)$. Find $h'(6)$ in terms of c.

(a) The average rate of change of f over the interval $[-5, 0]$ is $\frac{f(0) - f(-5)}{0 - (-5)} = \frac{2}{5}$.

1 : answer

(b) $g'(x) = f(x)$

The graph of g has a point of inflection at $x = -3$ because $g' = f$ changes from decreasing to increasing at this point.

The graph of g has a point of inflection at $x = 0$ because $g' = f$ changes from increasing to decreasing at this point.

3 : { 1 : $g'(x) = f(x)$; 1 : identifies $x = -3$ and $x = 0$; 1 : justification }

(c) $\frac{1}{c+5}\int_{-5}^{c} f(x)\,dx = \frac{1}{2}$

$\frac{1}{c+5}(-\pi + (-1) + 2 + c) = \frac{1}{2}$

$c = 3 + 2\pi$

3 : { 1 : integral; 1 : equation; 1 : answer }

(d) $h'(x) = \frac{1}{2} f'\left(\frac{x}{2}\right)$

$h'(6) = \frac{1}{2} f'(3) = \frac{1}{2} \cdot \frac{-2}{c} = -\frac{1}{c}$

2 : { 1 : $h'(x)$; 1 : answer }

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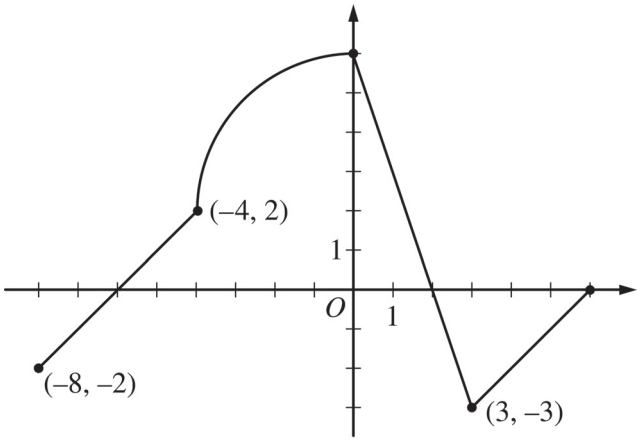

Graph of g

3. A continuous function g is defined on the closed interval $-8 \le x \le 6$. The graph of g, shown above, consists of three line segments and a quarter of a circle centered at the point $(0, 2)$. Let f be the function given by $f(x) = \int_{-8}^{x} g(t)\,dt$.

   (a) Find all values of x in the interval $-8 < x < 6$ at which f has a critical point. Classify each critical point as the location of a local minimum, a local maximum, or neither. Justify your answers.

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(b) Find $f(0)$.

(c) Find $\lim_{x \to -4} \dfrac{f(x)}{x^2 + 4x}$.

(d) Let h be the function defined by $h(x) = \dfrac{g(x)}{x^2 + 1}$. Find $h'(1)$.

GO ON TO THE NEXT PAGE.

Question 3

(a) $f'(x) = g(x)$

The function f has critical points at $x = -6$ and $x = 2$.

f has a local minimum at $x = -6$ because f' changes from negative to positive at that value.

f has a local maximum at $x = 2$ because f' changes from positive to negative at that value.

3 : { 1 : $f'(x) = g(x)$; 1 : critical points; 1 : classifications with justification }

(b) $f(0) = \int_{-8}^{0} g(t)\,dt = 8 + 4\pi$

1 : answer

(c) $\lim_{x \to -4} f(x) = f(-4) = \int_{-8}^{-4} g(t)\,dt = 0$

$\lim_{x \to -4} \left(x^2 + 4x\right) = 0$

Using L'Hospital's Rule,

$\lim_{x \to -4} \frac{f(x)}{x^2 + 4x} = \lim_{x \to -4} \frac{f'(x)}{2x + 4} = \lim_{x \to -4} \frac{g(x)}{2x + 4} = \frac{2}{-4} = -\frac{1}{2}$

2 : { 1 : L'Hospital's Rule; 1 : answer }

(d) $h'(x) = \frac{\left(x^2 + 1\right)g'(x) - g(x)2x}{\left(x^2 + 1\right)^2}$

$h'(1) = \frac{\left(1^2 + 1\right)g'(1) - g(1)(2)(1)}{\left(1^2 + 1\right)^2}$

$= \frac{2(-3) - 3(2)}{4} = -3$

3 : { 2 : $h'(x)$; 1 : answer }

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

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Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$

2 : $\frac{d^2y}{dx^2}$ in terms of x and y

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$

$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

2 : conclusion with justification

(c) $\lim_{x \to -1}(g(x) - 2) = 0$ and $\lim_{x \to -1} 3(x+1)^2 = 0$

Using L'Hospital's Rule,

$\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right) = \lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right)$

$\lim_{x \to -1} g'(x) = 0$ and $\lim_{x \to -1} 6(x+1) = 0$

Using L'Hospital's Rule,

$\lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x \to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$

3 : { 2 : L'Hospital's Rule; 1 : answer }

(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$

$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$

2 : { 1 : Euler's method; 1 : approximation }

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4. Let R be the region bounded above by the graph of $y = e^{-x}$, below by the x-axis, on the left by the y-axis, and on the right by the vertical line $x = m$, where $m > 0$ is a constant.

   (a) Find the area of R in terms of m.

   (b) Region R is revolved around the horizontal line $y = -3$ to form a solid. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

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Continue problem 4 on page 17.

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(c) Find the value of the real number $k > 1$ such that $\int_0^{\infty} k^{-x}\, dx = 2$, or show that no such k exists.

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Question 4

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(c) Find the value of the real number $k > 1$ such that $\int_0^{\infty} k^{-x}\,dx = 2$, or show that no such k exists.

(a) $\text{Area} = \int_0^m e^{-x}\,dx = -\left[e^{-x}\right]_0^m = -e^{-m} + 1$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $\text{Volume} = \pi \int_0^m \left[\left(e^{-x} + 3\right)^2 - 3^2\right] dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) $\lim_{b\to\infty} \int_0^b k^{-x}\,dx = -\frac{1}{\ln k} \lim_{b\to\infty} \left[k^{-x}\right]_0^b$

$= -\frac{1}{\ln k} \lim_{b\to\infty} \left(\frac{1}{k^b} - 1\right) = \frac{1}{\ln k}$

$\frac{1}{\ln k} = 2 \Rightarrow k = e^{1/2}$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit} \\ 1 : \text{answer} \end{cases}$

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4. Consider the differential equation $\frac{dy}{dx} = (y - 2)(x^2 + 1)$.

(a) Find $y = g(x)$, the particular solution to the given differential equation with initial condition $g(0) = 5$.

(b) For the particular solution $y = g(x)$ found in part (a), find $\lim_{x \to -\infty} g(x)$.

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(c) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 3$.

Find the value of $\frac{d^2y}{dx^2}$ at the point $(1, 3)$. Is the graph of $y = f(x)$ concave up or concave down at the point $(1, 3)$? Give a reason for your answer.

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Question 4

(a) $\dfrac{dy}{dx} = (y-2)\left(x^2+1\right)$

$\displaystyle\int \frac{dy}{y-2} = \int \left(x^2+1\right) dx$

$\ln|y-2| = \dfrac{x^3}{3} + x + C$

$\ln 3 = \dfrac{0^3}{3} + 0 + C \Rightarrow C = \ln 3$

Because $y(0) = 5,\ y > 2,$ so $|y-2| = y-2.$

$y - 2 = 3e^{\frac{x^3}{3}+x}$

$y = 2 + 3e^{\frac{x^3}{3}+x}$

Note: this solution is valid for all real numbers.

5 : { 1 : separation of variables; 2 : antiderivatives; 1 : constant of integration and uses initial condition; 1 : solves for y }

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\displaystyle\lim_{x\to-\infty}\left(2 + 3e^{\frac{x^3}{3}+x}\right) = 2$

1 : answer

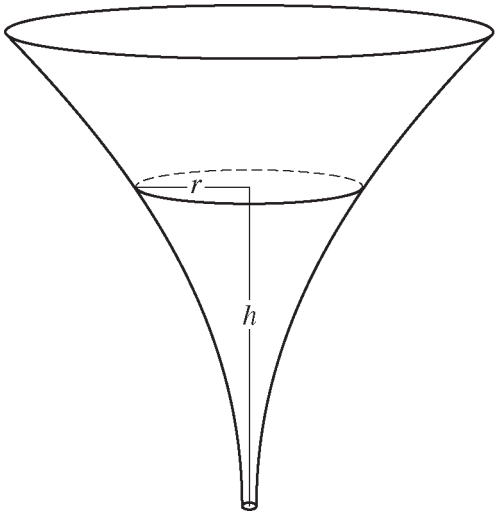
(c) $\left.\dfrac{dy}{dx}\right|_{(1,\,3)} = (3-2)\left(1^2+1\right) = 2$

$\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}\left(x^2+1\right) + (y-2)(2x)$

$\left.\dfrac{d^2y}{dx^2}\right|_{(1,\,3)} = (2)\left(1^2+1\right) + (3-2)(2) = 6$

Because $\left.\dfrac{d^2y}{dx^2}\right|_{(1,\,3)} > 0$ and $\dfrac{d^2y}{dx^2}$ is continuous, the graph of $y = f(x)$ is concave up at the point $(1, 3)$.

3 : { 2 : $\left.\dfrac{d^2y}{dx^2}\right|_{(1,\,3)}$; 1 : concave up with reason }



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

   (a) Find the average value of the radius of the funnel.

   (b) Find the volume of the funnel.

   (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

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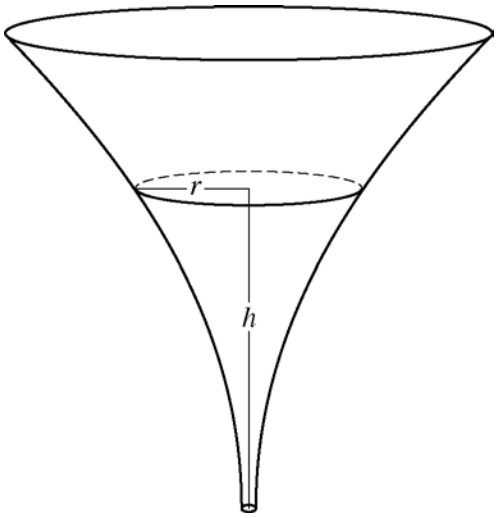
Question 5



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(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right)\,dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60}$ in

3 : { 1 : integral; 1 : antiderivative; 1 : answer }

(b) Volume $= \pi\int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2\,dh = \frac{\pi}{400}\int_0^{10}\left(9 + 6h^2 + h^4\right)\,dh$

$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$

$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40}$ in^3

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{1}{5}\cdot\frac{10}{3} = -\frac{2}{3}$ in/sec

3 : { 2 : chain rule; 1 : answer }

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5. For $0 \le t \le 24$ hours, the temperature inside a refrigerator in a kitchen is given by the function W that satisfies the differential equation $\frac{dW}{dt} = \frac{3\cos t}{2W}$. $W(t)$ is measured in degrees Celsius (°C), and t is measured in hours. At time $t = 0$ hours, the temperature inside the refrigerator is 3°C.

   (a) Write an equation for the line tangent to the graph of $y = W(t)$ at the point where $t = 0$. Use the equation to approximate the temperature inside the refrigerator at $t = 0.4$ hour.

   (b) Find $y = W(t)$, the particular solution to the differential equation with initial condition $W(0) = 3$.

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Continue problem 5 on page 19.

(c) The temperature in the kitchen remains constant at 20°C for $0 \le t \le 24$. The cost of operating the refrigerator accumulates at the rate of \$0.001 per hour for each degree that the temperature in the kitchen exceeds the temperature inside the refrigerator. Write, but do not evaluate, an expression involving an integral that can be used to find the cost of operating the refrigerator for the 24-hour interval.

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Question 5

For $0 \le t \le 24$ hours, the temperature inside a refrigerator in a kitchen is given by the function W that satisfies the differential equation $\frac{dW}{dt} = \frac{3\cos t}{2W}$. $W(t)$ is measured in degrees Celsius (°C), and t is measured in hours. At time $t = 0$ hours, the temperature inside the refrigerator is 3°C.

(a) Write an equation for the line tangent to the graph of $y = W(t)$ at the point where $t = 0$. Use the equation to approximate the temperature inside the refrigerator at $t = 0.4$ hour.

(b) Find $y = W(t)$, the particular solution to the differential equation with initial condition $W(0) = 3$.

(c) The temperature in the kitchen remains constant at 20°C for $0 \le t \le 24$. The cost of operating the refrigerator accumulates at the rate of \$0.001 per hour for each degree that the temperature in the kitchen exceeds the temperature inside the refrigerator. Write, but do not evaluate, an expression involving an integral that can be used to find the cost of operating the refrigerator for the 24-hour interval.

(a) $\left.\frac{dW}{dt}\right|_{(t, W)=(0, 3)} = \frac{3\cos 0}{2(3)} = \frac{1}{2}$

An equation for the tangent line is $y = \frac{1}{2}t + 3$.

$W(0.4) \approx \frac{1}{2}(0.4) + 3 = 3.2$°C

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(b) $2W\,dW = 3\cos t\,dt$

$\int 2W\,dW = \int 3\cos t\,dt$

$W^2 = 3\sin t + C$

$3^2 = 3\sin 0 + C \Rightarrow C = 9$

$W^2 = 3\sin t + 9$

Since $W(0) = 3$, $W = \sqrt{3\sin t + 9}$ for $0 \le t \le 24$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(c) $0.001\int_0^{24}(20 - W(t))\,dt$ dollars

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

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x	1	2	3	4	5
$f(x)$	4	8	7	3	–1
$f'(x)$	5	–2	–1	6	2
$f''(x)$	2	–3	1	–1	3

5. The table above gives values of a function f, its first derivative f', and its second derivative f'' for selected values of x. The function f'' is continuous for all real numbers.

   (a) Let g be the function given by $g(x) = (2x - 1)f(x)$. Find $g'(3)$.

   (b) Let h be the function given by $h(x) = f(f(x))$. Find $h'(4)$.

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(c) Find the value of $\int_1^5 x f''(x)\, dx$.

(d) Is there a value c, for $3 < c < 4$, such that $f''(c) = 7$? Justify your answer.

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Question 5

(a) $g'(x) = 2f(x) + (2x-1)f'(x)$

$g'(3) = 2f(3) + 5f'(3)$
$= (2)(7) + (5)(-1) = 14 - 5 = 9$

2 : $\begin{cases} 1 : g'(x) \\ 1 : g'(3) \end{cases}$

(b) $h'(x) = f'(f(x)) \cdot f'(x)$

$h'(4) = f'(f(4)) \cdot f'(4)$
$= f'(3) \cdot 6 = (-1)(6) = -6$

2 : $\begin{cases} 1 : h'(x) \\ 1 : h'(4) \end{cases}$

(c) $u = x \quad dv = f''(x)\,dx$
$du = dx \quad v = f'(x)$

$\int x f''(x)\,dx = xf'(x) - \int f'(x)\,dx$
$= xf'(x) - f(x)$

$\int_1^5 x f''(x)\,dx = [xf'(x) - f(x)]_1^5$
$= (5f'(5) - f(5)) - (f'(1) - f(1))$
$= (5 \cdot 2 - (-1)) - (5 - 4) = 10$

3 : $\begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) f'' is continuous. $\Rightarrow$ f' is differentiable and continuous on $3 \le x \le 4$.

$\dfrac{f'(4) - f'(3)}{4 - 3} = \dfrac{6 - (-1)}{1} = 7$

Therefore, by the Mean Value Theorem, there is a value c, for $3 < c < 4$, such that $f''(c) = 7$.

2 : $\begin{cases} 1 : \dfrac{f'(4) - f'(3)}{4 - 3} \\ 1 : \text{conclusion, using MVT} \end{cases}$

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

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Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

4 : { 1 : first two terms; 1 : third term; 1 : fourth term; 1 : general term }

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

2 : { 1 : identifies both endpoints; 1 : analysis and interval of convergence }

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$

2 : { 1 : error form; 1 : analysis }

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6. Let f be a function that has derivatives of all orders for all real numbers, and let $3 - 6(x-1)^2 + 5(x-1)^3 - 2(x-1)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 1$.

(a) Find $f'''(1)$.

(b) Does the graph of f have a relative maximum, a relative minimum, or neither at $x = 1$? Give a reason for your answer.

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Continue problem 6 on page 21.

(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$. Find the third-degree Taylor polynomial for g about $x = 1$, and use it to approximate $g(1.1)$.

(d) Let h be the function defined by $h(x) = f'(x)$. Find the third-degree Taylor polynomial for h about $x = 1$. Verify that the graph of h does not have a point of inflection at $x = 1$.

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Question 6

Let f be a function that has derivatives of all orders for all real numbers, and let $3 - 6(x-1)^2 + 5(x-1)^3 - 2(x-1)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 1$.

(a) Find $f'''(1)$.

(b) Does the graph of f have a relative maximum, a relative minimum, or neither at $x = 1$? Give a reason for your answer.

(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$. Find the third-degree Taylor polynomial for g about $x = 1$, and use it to approximate $g(1.1)$.

(d) Let h be the function defined by $h(x) = f'(x)$. Find the third-degree Taylor polynomial for h about $x = 1$. Verify that the graph of h does not have a point of inflection at $x = 1$.

(a) $5 = \dfrac{f'''(1)}{6} \Rightarrow f'''(1) = 30$

1 : answer

(b) $f'(1) = 0$

$-6 = \dfrac{f''(1)}{2} \Rightarrow f''(1) = -12 < 0$

Therefore, f has a relative maximum at $x = 1$.

2 : { 1 : $f'(1) = 0$; 1 : answer with reason }

(c) Let G_3 be the third-degree Taylor polynomial for g about $x = 1$.

$$G_3(x) = \int_1^x \left(3 - 6(t-1)^2\right) dt$$
$$= \left[3(t-1) - 2(t-1)^3\right]_1^x$$
$$= 3(x-1) - 2(x-1)^3$$

$g(1.1) \approx G_3(1.1) = 3(0.1) - 2(0.1)^3 = 0.298$

3 : { 2 : Taylor polynomial; 1 : approximation }

(d) Let H_3 be the third-degree Taylor polynomial for h about $x = 1$.

$$H_3(x) = \frac{d}{dx}\left[3 - 6(x-1)^2 + 5(x-1)^3 - 2(x-1)^4\right]$$
$$= -12(x-1) + 15(x-1)^2 - 8(x-1)^3$$

$15 = \dfrac{h''(1)}{2} \Rightarrow h''(1) = 30$

Since $h''(1) \neq 0$, the graph of h does not have a point of inflection at $x = 1$.

3 : { 2 : Taylor polynomial; 1 : reason }

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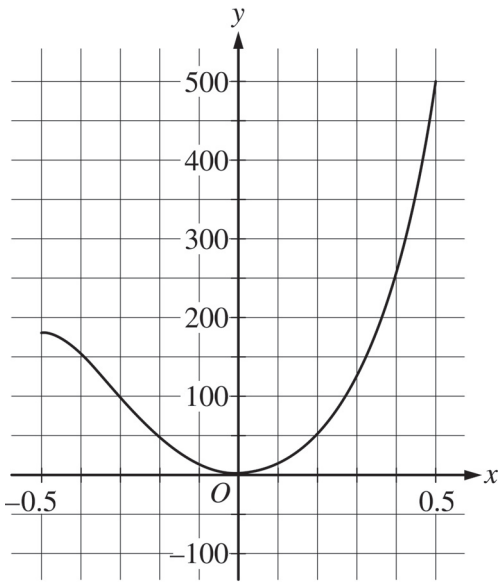

Graph of $f^{(5)}$

6. Let f and g be the functions given by $f(x) = xe^{x^3}$ and $g(x) = \int_0^x f(t)\,dt$. The graph of $f^{(5)}$, the fifth derivative of f, is shown above for $-\frac{1}{2} \le x \le \frac{1}{2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for e^x about $x = 0$.
   Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

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(b) Write the first four nonzero terms of the Taylor series for g about $x = 0$.

(c) Find the value of $g^{(5)}(0)$.

(d) Let $P_5(x)$ be the fifth-degree Taylor polynomial for g about $x = 0$. Use the Lagrange error bound along with information from the given graph to find an upper bound on $\left| P_5\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$.

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Question 6

(a) The Taylor series for e^x about $x = 0$ is
$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$.

First four nonzero terms for f :
$x\left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!}\right) = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!}$

General term for f : $\frac{x^{3n+1}}{n!}$

4 : { 1 : terms for e^x ; 2 : first four terms for f ; 1 : general term for f }

(b) $\int_0^x f(t)\,dt = \left[\frac{t^2}{2} + \frac{t^5}{5} + \frac{t^8}{16} + \frac{t^{11}}{66} + \cdots\right]_0^x$
$= \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{16} + \frac{x^{11}}{66} + \cdots$

First four nonzero terms for g :
$\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{16} + \frac{x^{11}}{66}$

2 : { 1 : two terms ; 1 : remaining terms }

(c) $\frac{g^{(5)}(0)}{5!} = \frac{1}{5} \Rightarrow g^{(5)}(0) = \frac{5!}{5} = 24$

1 : answer

(d) $\left|P_5\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| \le \frac{\max\limits_{0 \le x \le 1/2} \left|g^{(6)}(x)\right|}{6!}\left(\frac{1}{2}\right)^6$
$= \frac{\max\limits_{0 \le x \le 1/2} \left|f^{(5)}(x)\right|}{6!\,2^6} \le \frac{500}{6!\,2^6}$

2 : { 1 : form of the error bound ; 1 : upper bound }

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

   (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

   (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

   (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

   (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

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Question 1

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1 : units in parts (a), (c), and (d)

(a) Volume $= \int_0^{10} A(h)\,dh$

$\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$

$= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$

$= 176.3$ cubic feet

2 : { 1 : left Riemann sum; 1 : approximation }

(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.

1 : overestimate with reason

(c) $\int_0^{10} f(h)\,dh = 101.325338$

The volume is 101.325 cubic feet.

2 : { 1 : integral; 1 : answer }

(d) Using the model, $V(h) = \int_0^h f(x)\,dx$.

$$\left.\frac{dV}{dt}\right|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$$

$$= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$$

$$= f(5)\cdot 0.26 = 1.694419$$

When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.

3 : { 2 : $\frac{dV}{dt}$; 1 : answer }

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1. For $0 \le t \le 8$, a particle moving in the xy-plane has position vector $\langle x(t), y(t) \rangle = \langle \sin(2t), t^2 - t \rangle$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds.

   (a) Find the speed of the particle at time $t = 2$ seconds. Indicate units of measure.

   (b) At time $t = 4$ seconds, is the speed of the particle increasing or decreasing? Explain your answer.

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Continue question 1 on page 5.

(c) Find the total distance the particle travels over the time interval $0 \le t \le 5$ seconds.

(d) At time $t = 8$ seconds, the particle begins moving in a straight line. For $t \ge 8$, the particle travels with the same velocity vector that it had at time $t = 8$ seconds. Find the position of the particle at time $t = 10$ seconds.

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Question 1

(a) $\sqrt{(x'(2))^2 + (y'(2))^2} = 3.272461$

The speed of the particle at time $t = 2$ seconds is 3.272 meters per second.

$2 : \begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer with units} \end{cases}$

(b) $s(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(2\cos(2t))^2 + (2t-1)^2}$

$s'(4) = 2.16265$

Since $s'(4) > 0$, the speed of the particle is increasing at time $t = 4$.

$2 : \begin{cases} 1 : \text{considers } s'(4) \\ 1 : \text{answer with reason} \end{cases}$

(c) $\int_0^5 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 22.381767$

The total distance the particle travels over the time interval $0 \le t \le 5$ seconds is 22.382 (or 22.381) meters.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $x(10) = x(8) + x'(8) \cdot 2 = \sin 16 + x'(8) \cdot 2 = -4.118541$

$y(10) = y(8) + y'(8) \cdot 2 = \left(8^2 - 8\right) + y'(8) \cdot 2 = 86$

The position of the particle at time $t = 10$ seconds is $(-4.119, 86)$ (or $(-4.118, 86)$).

$3 : \begin{cases} 1 : \text{uses position at } t = 8 \\ 1 : \text{uses velocity at } t = 8 \\ 1 : \text{position at } t = 10 \end{cases}$

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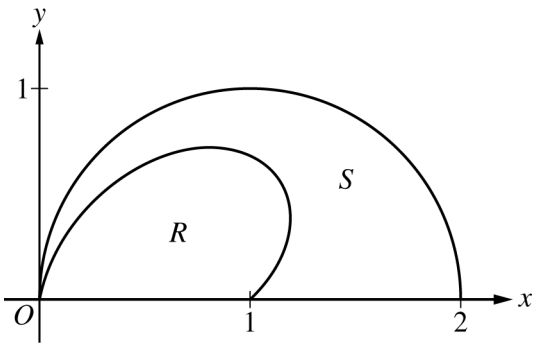

2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

END OF PART A OF SECTION II

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Question 2

(a) $\frac{1}{2}\int_0^{\pi/2}(f(\theta))^2\,d\theta = 0.648414$

The area of R is 0.648.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$

— OR —

$\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \int_k^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$

$2 : \begin{cases} 1 : \text{integral expression for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$w_A = \dfrac{\int_0^{\pi/2} w(\theta)\,d\theta}{\frac{\pi}{2}-0} = 0.485446$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

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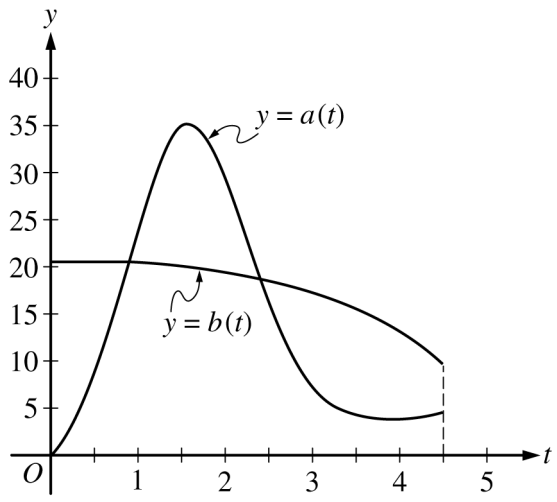

2. During the time interval $0 \le t \le 4.5$ hours, water flows into tank A at a rate of $a(t) = (2t - 5) + 5e^{2 \sin t}$ liters per hour. During the same time interval, water flows into tank B at a rate of $b(t)$ liters per hour. Both tanks are empty at time $t = 0$. The graphs of $y = a(t)$ and $y = b(t)$, shown in the figure above, intersect at $t = k$ and $t = 2.416$.

(a) How much water will be in tank A at time $t = 4.5$?

(b) During the time interval $0 \le t \le k$ hours, water flows into tank B at a constant rate of 20.5 liters per hour. What is the difference between the amount of water in tank A and the amount of water in tank B at time $t = k$?

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Continue question 2 on page 7.

(c) The area of the region bounded by the graphs of $y = a(t)$ and $y = b(t)$ for $k \le t \le 2.416$ is 14.470. How much water is in tank B at time $t = 2.416$?

(d) During the time interval $2.7 \le t \le 4.5$ hours, the rate at which water flows into tank B is modeled by $w(t) = 21 - \dfrac{30t}{(t-8)^2}$ liters per hour. Is the difference $w(t) - a(t)$ increasing or decreasing at time $t = 3.5$? Show the work that leads to your answer.

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Question 2

(a) $\int_0^{4.5} a(t)\,dt = 66.532128$

At time $t = 4.5$, tank A contains 66.532 liters of water.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $a(k) = 20.5 \Rightarrow k = 0.892040$

$\int_0^k (20.5 - a(t))\,dt = 10.599191$

At time $t = k$, the difference in the amounts of water in the tanks is 10.599 liters.

$3 : \begin{cases} 1 : \text{sets } a(k) = 20.5 \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{2.416} b(t)\,dt = \int_0^k b(t)\,dt + \int_k^{2.416} b(t)\,dt$

$\int_0^k b(t)\,dt = 20.5 \cdot k = 18.286826$

On $k < t < 2.416$, tank A receives $\int_k^{2.416} a(t)\,dt = 44.497051$ liters of water, which is 14.470 more liters of water than tank B.
Therefore, $\int_k^{2.416} b(t)\,dt = \int_k^{2.416} a(t)\,dt - 14.470 = 30.027051$.

$\int_0^k b(t)\,dt + \int_k^{2.416} b(t)\,dt = 48.313876$

At time $t = 2.416$, tank B contains 48.314 (or 48.313) liters of water.

$2 : \begin{cases} 1 : \int_k^{2.416} a(t)\,dt \\ 1 : \text{answer} \end{cases}$

(d) $w'(3.5) - a'(3.5) = -1.14298 < 0$

The difference $w(t) - a(t)$ is decreasing at $t = 3.5$.

$2 : \begin{cases} 1 : w'(3.5) - a'(3.5) < 0 \\ 1 : \text{conclusion} \end{cases}$

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CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

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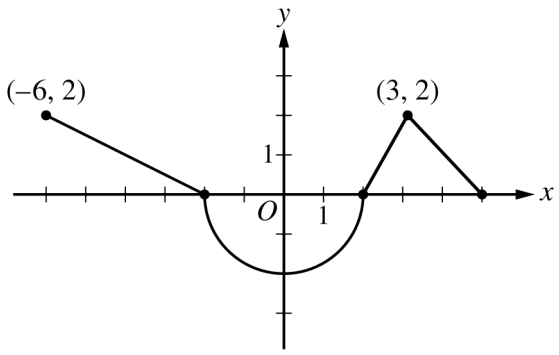

Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find the values of $f(-6)$ and $f(5)$.

   (b) On what intervals is f increasing? Justify your answer.

   (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

   (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

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Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

3 : $\begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

2 : $\begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}$.

2 : $\begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist, with explanation} \end{cases}$

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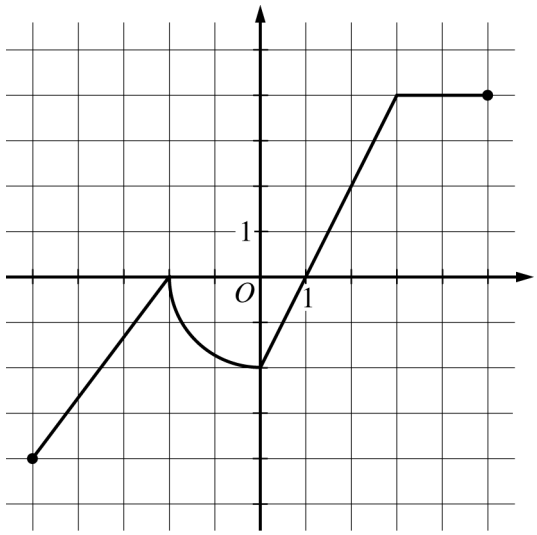

Graph of f

3. The graph of the function f, consisting of three line segments and a quarter of a circle, is shown above. Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the average rate of change of g from $x = -5$ to $x = 5$.

(b) Find the instantaneous rate of change of g with respect to x at $x = 3$, or state that it does not exist.

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Continue question 3 on page 15.

(c) On what open intervals, if any, is the graph of g concave up? Justify your answer.

(d) Find all x-values in the interval $-5 < x < 5$ at which g has a critical point. Classify each critical point as the location of a local minimum, a local maximum, or neither. Justify your answers.

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Question 3

(a) $\frac{g(5)-g(-5)}{5-(-5)} = \frac{12-(\pi+7)}{10} = \frac{5-\pi}{10}$	3 : $\begin{cases} 1 : \text{difference quotient} \\ 2 : \text{answer} \end{cases}$
(b) $g'(x) = f(x)$ $g'(3) = f(3) = 4$ The instantaneous rate of change of g at $x = 3$ is 4.	1 : answer
(c) The graph of g is concave up on $-5 < x < -2$ and $0 < x < 3$, because $g'(x) = f(x)$ is increasing on these intervals.	2 : intervals with justification
(d) $g'(x) = f(x)$ is defined at all x with $-5 < x < 5$. $g'(x) = f(x) = 0$ at $x = -2$ and $x = 1$. Therefore, g has critical points at $x = -2$ and $x = 1$. g has neither a local maximum nor a local minimum at $x = -2$ because g' does not change sign there. g has a local minimum at $x = 1$ because g' changes from negative to positive there.	3 : $\begin{cases} 1 : \text{considers } f(x) = 0 \\ 1 : \text{critical points at } x = -2 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

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4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

   (a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

   (b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

   (c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

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Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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x	0	1	2	3	4	5	6
$f'(x)$	4	3.5	2	0.8	1.7	5.8	7

4. The function f satisfies $f(0) = 20$. The first derivative of f satisfies the inequality $0 \le f'(x) \le 7$ for all x in the closed interval $[0, 6]$. Selected values of f' are shown in the table above. The function f has a continuous second derivative for all real numbers.

   (a) Use a midpoint Riemann sum with three subintervals of equal length indicated by the data in the table to approximate the value of $f(6)$.

   (b) Determine whether the actual value of $f(6)$ could be 70. Explain your reasoning.

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Continue question 4 on page 17.

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(c) Evaluate $\int_2^4 f''(x)\,dx$.

(d) Find $\lim_{x \to 0} \dfrac{f(x) - 20e^x}{0.5f(x) - 10}$.

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Question 4

(a) $\int_0^6 f'(x)\,dx \approx 2 \cdot 3.5 + 2 \cdot 0.8 + 2 \cdot 5.8 = 20.2$

$f(6) - f(0) = \int_0^6 f'(x)\,dx$

$f(6) = f(0) + \int_0^6 f'(x)\,dx \approx 20 + 20.2 = 40.2$

3 : { 1 : midpoint sum; 1 : Fundamental Theorem of Calculus; 1 : answer }

(b) Since $f'(x) \le 7$, $\int_0^6 f'(x)\,dx \le 6 \cdot 7 = 42$.

$f(6) - f(0) \le 42 \Rightarrow f(6) \le 20 + 42 = 62$

Therefore, the actual value of $f(6)$ could not be 70.

2 : { 1 : integral bound; 1 : answer with reasoning }

(c) $\int_2^4 f''(x)\,dx = f'(4) - f'(2) = 1.7 - 2 = -0.3$

2 : { 1 : Fundamental Theorem of Calculus; 1 : answer }

(d) $\lim_{x \to 0} \left(f(x) - 20e^x \right) = 0$

$\lim_{x \to 0} (0.5f(x) - 10) = 0$

Using L'Hospital's Rule,

$\lim_{x \to 0} \frac{f(x) - 20e^x}{0.5f(x) - 10} = \lim_{x \to 0} \frac{f'(x) - 20e^x}{0.5f'(x)} = \frac{4 - 20}{0.5(4)} = -8$

2 : { 1 : L'Hospital's Rule; 1 : answer }

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5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

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Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2}$

$f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$

2 : $f'(3)$

(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$

The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\dfrac{7}{4}$.

f' changes sign from positive to negative at $x = \dfrac{7}{4}$.

Therefore, f has a relative maximum at $x = \dfrac{7}{4}$.

2 : { 1 : x-coordinate; 1 : relative maximum with justification

(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$

$\displaystyle= \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$

$\displaystyle= \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$

3 : { 1 : antiderivative; 1 : limit expression; 1 : answer

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges.

— OR —

$\dfrac{3}{2n^2-7n+5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \geq 5$.

Since $\displaystyle\lim_{n\to\infty}\frac{\dfrac{3}{2n^2-7n+5}}{\dfrac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\displaystyle\sum_{n=5}^\infty \frac{3}{2n^2-7n+5}$ converges by the limit comparison test.

2 : answer with conditions

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5. Consider the differential equation $\frac{dy}{dx} = -1 + \frac{y^2}{x}$.

(a) Show that $\frac{d^2y}{dx^2} = \frac{2y^3 - y^2 - 2xy}{x^2}$.

(b) Let $y = g(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = -1 + \frac{y^2}{x}$ with initial condition $g(4) = 2$. Does g have a relative minimum, a relative maximum, or neither at $x = 4$? Justify your answer.

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Continue question 5 on page 19.

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(c) Let $y = h(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = -1 + \frac{y^2}{x}$ with initial condition $h(1) = 2$. Write the second-degree Taylor polynomial for h about $x = 1$.

(d) For the function h given in part (c), it is known that $\left|h'''(x)\right| \leq 60$ for all x in the interval $0.9 \leq x \leq 1.1$. Let A represent the approximation of $h(1.1)$ found by using the second-degree Taylor polynomial for h about $x = 1$ from part (c). Use the Lagrange error bound to show that A differs from $h(1.1)$ by at most 0.01.

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Question 5

(a) $\dfrac{d^2y}{dx^2} = \dfrac{x \cdot 2y\dfrac{dy}{dx} - y^2 \cdot 1}{x^2}$

$= \dfrac{2xy\left(-1 + \dfrac{y^2}{x}\right) - y^2}{x^2} = \dfrac{2y^3 - y^2 - 2xy}{x^2}$

$2 : \begin{cases} 1 : \text{quotient rule} \\ 1 : \dfrac{d^2y}{dx^2} \end{cases}$

(b) $\left.\dfrac{dy}{dx}\right|_{(x,y)=(4,\,2)} = -1 + \dfrac{4}{4} = 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(4,\,2)} = \dfrac{2 \cdot 8 - 4 - 16}{16} = -\dfrac{1}{4} < 0$

By the Second Derivative Test, g has a relative maximum at $x = 4$.

$2 : \begin{cases} 1 : \text{considers } \left.\dfrac{dy}{dx}\right|_{(x,y)=(4,\,2)} \\ 1 : \text{answer with justification} \end{cases}$

(c) $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,\,2)} = -1 + \dfrac{4}{1} = 3$

$\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(1,\,2)} = \dfrac{2 \cdot 8 - 4 - 4}{1} = 8$

The second-degree Taylor polynomial for h about $x = 1$ is

$T_2(x) = 2 + 3(x-1) + \dfrac{8}{2!}(x-1)^2 = 2 + 3(x-1) + 4(x-1)^2$.

$3 : \begin{cases} 1 : \text{uses } \left.\dfrac{dy}{dx}\right|_{(x,\,y)=(1,\,2)} \text{ and } \left.\dfrac{d^2y}{dx^2}\right|_{(x,\,y)=(1,\,2)} \\ 2 : \text{Taylor polynomial} \end{cases}$

(d) $|h(1.1) - A| \le \dfrac{\max\limits_{1.0 \le x \le 1.1} |h'''(x)|\,|1.1-1|^3}{3!} \le \dfrac{60}{6} \cdot \dfrac{1}{1000} = \dfrac{10}{1000} = \dfrac{1}{100}$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

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$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

   (a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

   (b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

   (c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

   (d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$.

STOP

END OF EXAM

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Question 6

(a) $f(0) = 0$
$f'(0) = 1$
$f''(0) = -1(1) = -1$
$f'''(0) = -2(-1) = 2$
$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are
$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$.

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

3 : { 1 : $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$; 1 : verify terms; 1 : general term }

(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0 and therefore the series converges conditionally.

2 : converges conditionally with reason

(c) $\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$

$= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$

$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$

3 : { 1 : two terms; 1 : remaining terms; 1 : general term }

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}$.

1 : error bound

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6. Let f be the function defined by $f(x) = \dfrac{1}{x^2 + 9}$.

(a) Evaluate the improper integral $\displaystyle\int_3^{\infty} f(x)\,dx$, or show that the integral diverges.

(b) Determine whether the series $\displaystyle\sum_{n=3}^{\infty} f(n)$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

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Continue question 6 on page 21.

(c) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(e^n \cdot f(n)\right)} = \sum_{n=1}^{\infty} \frac{(-1)^n\left(n^2+9\right)}{e^n}$ converges absolutely, converges conditionally, or diverges.

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Question 6

(a) $\displaystyle\int_3^\infty \frac{1}{x^2+9}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{x^2+9}\,dx = \lim_{b\to\infty}\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_3^b\right)$

$\displaystyle= \lim_{b\to\infty}\left(\frac{1}{3}\tan^{-1}\left(\frac{b}{3}\right) - \frac{1}{3}\tan^{-1}(1)\right) = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$

3 : { 1 : antiderivative; 1 : limit expression; 1 : answer }

(b) The function f is continuous, positive, and decreasing on $[3, \infty)$.

By the integral test, since $\displaystyle\int_3^\infty f(x)\,dx$ converges, $\displaystyle\sum_{n=3}^\infty f(n)$ converges.

— OR —

$\displaystyle 0 < \frac{1}{n^2+9} < \frac{1}{n^2}$ for $n \ge 3$.

Since the series $\displaystyle\sum_{n=3}^\infty \frac{1}{n^2}$ converges, the series $\displaystyle\sum_{n=3}^\infty f(n) = \sum_{n=3}^\infty \frac{1}{n^2+9}$ converges by the comparison test.

2 : conclusion with conditions

(c) Consider the series $\displaystyle\sum_{n=1}^\infty \frac{1}{\left(e^n \cdot f(n)\right)} = \sum_{n=1}^\infty \frac{n^2+9}{e^n}$.

$\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2+9}{e^{n+1}}}{\frac{n^2+9}{e^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2+9}{n^2+9}\cdot\frac{1}{e}\right| = \frac{1}{e} < 1$

By the ratio test, $\displaystyle\sum_{n=1}^\infty \frac{1}{\left(e^n \cdot f(n)\right)}$ converges.

Therefore, $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{\left(e^n \cdot f(n)\right)}$ converges absolutely.

4 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : conclusion of ratio test; 1 : converges absolutely }

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CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

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Question 1

(a) $\int_0^{300} r(t)\,dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \le t \le 300$.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $20 + \int_0^{300} (r(t) - 0.7)\,dt = 20 + \int_0^{300} r(t)\,dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

$2 : \begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time t that there are no people in line is $300 + \frac{80}{0.7} = 414.286$ (or 414.285) seconds.

1 : answer

(d) The total number of people in line at time t, $0 \le t \le 300$, is modeled by $20 + \int_0^t r(x)\,dx - 0.7t$.

$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298,\ t_2 = 166.574719$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

$4 : \begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

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t (minutes)	0	1	5	6	8
$g(t)$ (cubic feet per minute)	12.8	15.1	20.5	18.3	22.7

1. Grain is being added to a silo. At time $t = 0$, the silo is empty. The rate at which grain is being added is modeled by the differentiable function g, where $g(t)$ is measured in cubic feet per minute for $0 \le t \le 8$ minutes. Selected values of $g(t)$ are given in the table above.

   (a) Using the data in the table, approximate $g'(3)$. Using correct units, interpret the meaning of $g'(3)$ in the context of the problem.

   (b) Write an integral expression that represents the total amount of grain added to the silo from time $t = 0$ to time $t = 8$. Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate the integral.

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Continue question 1 on page 5.

(c) The grain in the silo is spoiling at a rate modeled by $w(t) = 32 \cdot \sqrt{\sin\left(\frac{\pi t}{74}\right)}$, where $w(t)$ is measured in cubic feet per minute for $0 \le t \le 8$ minutes. Using the result from part (b), approximate the amount of unspoiled grain remaining in the silo at time $t = 8$.

(d) Based on the model in part (c), is the amount of unspoiled grain in the silo increasing or decreasing at time $t = 6$? Show the work that leads to your answer.

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Question 1

(a) $g'(3) \approx \dfrac{g(5) - g(1)}{5 - 1} = \dfrac{20.5 - 15.1}{4} = 1.35$

At time $t = 3$ minutes, the rate at which grain is being added to the silo is increasing at a rate of 1.35 cubic feet per minute per minute.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{interpretation with units} \end{cases}$

(b) The total amount of grain added to the silo from time $t = 0$ to time $t = 8$ is $\int_0^8 g(t)\,dt$ cubic feet.

$$\int_0^8 g(t)\,dt \approx g(1)\cdot(1-0) + g(5)\cdot(5-1) + g(6)\cdot(6-5) + g(8)\cdot(8-6)$$
$$= 15.1\cdot 1 + 20.5\cdot 4 + 18.3\cdot 1 + 22.7\cdot 2 = 160.8$$

$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

(c) $\int_0^8 w(t)\,dt = 99.051497$

The approximate amount of unspoiled grain remaining in the silo at time $t = 8$ is $160.8 - \int_0^8 w(t)\,dt = 61.749$ (or 61.748) cubic feet.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $g(6) - w(6) = 18.3 - 16.063173 = 2.236827 > 0$

Because $g(6) - w(6) > 0$, the amount of unspoiled grain is increasing at time $t = 6$.

$2 : \begin{cases} 1 : \text{considers } g(6) - w(6) \\ 1 : \text{answer} \end{cases}$

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2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

(c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\,dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

(d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II

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Question 2

(a) $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter.

2 : { 1 : answer; 1 : meaning with units }

(b) $\int_0^{30} 3p(h)\,dh = 1675.414936$

There are 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters.

2 : { 1 : integrand; 1 : answer }

(c) $\int_{30}^{K} 3f(h)\,dh$ represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of K meters.

The number of plankton cells, in millions, in the entire column of water is given by $\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh$.

Because $0 \le f(h) \le u(h)$ for all $h \ge 30$,

$3\int_{30}^{K} f(h)\,dh \le 3\int_{30}^{K} u(h)\,dh \le 3\int_{30}^{\infty} u(h)\,dh = 3 \cdot 105 = 315.$

The total number of plankton cells in the column of water is bounded by $1675.415 + 315 = 1990.415 \le 2000$ million.

3 : { 1 : integral expression; 1 : compares improper integral; 1 : explanation }

(d) $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = 757.455862$

The total distance traveled by the boat over the time interval $0 \le t \le 1$ is 757.456 (or 757.455) meters.

2 : { 1 : integrand; 1 : total distance }

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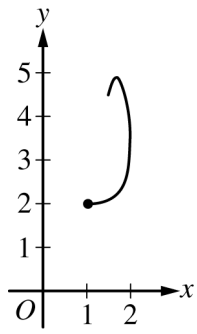

2. A particle moving in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$, where $\frac{dx}{dt} = \cos\left(t^2\right)$ and $\frac{dy}{dt} = e^t \sin\left(t^2\right)$. At time $t = 0$, the particle is at position $(1, 2)$. The figure above shows the path of the particle for $0 \leq t \leq 2$.

(a) Find the position of the particle at time $t = 2$.

(b) Find the slope of the line tangent to the particle's path at time $t = 2$.

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Continue question 2 on page 7.

(c) Find the speed of the particle at time $t = 2$. Find the acceleration vector of the particle at time $t = 2$.

(d) Consider a rectangle with vertices at points $(0, 0)$, $(x(t), 0)$, $(x(t), y(t))$, and $(0, y(t))$ at time $t \geq 0$. For $0 \leq t \leq 2$, at what time t is the perimeter of the rectangle a maximum? Justify your answer.

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Question 2

(a) $x(2) = 1 + \int_0^2 \frac{dx}{dt}\,dt = 1.461461$

$y(2) = 2 + \int_0^2 \frac{dy}{dt}\,dt = 4.268236$

The position of the particle at time $t = 2$ is $(1.461, 4.268)$.

3 : { 1 : integrals; 1 : uses initial condition; 1 : position }

(b) $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{dy/dt}{dx/dt}\right|_{t=2} = 8.555206$

The slope of the line tangent to the particle's path at time $t = 2$ is 8.555.

1 : slope

(c) $\left.\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right|_{t=2} = 5.630128$

The speed of the particle at time $t = 2$ is 5.630.

$x''(2) = 3.027210$

$y''(2) = -24.911294$

The acceleration vector of the particle at time $t = 2$ is $\langle 3.027, -24.911 \rangle$.

2 : { 1 : speed; 1 : acceleration vector }

(d) Perimeter $= p(t) = 2(x(t) + y(t))$

$p'(t) = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 0 \Rightarrow t = 1.721837$

Because $t = 1.721837$ is the only interior critical point for $0 \le t \le 2$ and $p'(t)$ changes from positive to negative at this point, the perimeter of the rectangle is a maximum at $t = 1.722$ (or $t = 1.721$).

3 : { 1 : sets $p'(t) = 0$; 2 : answer with justification }

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CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

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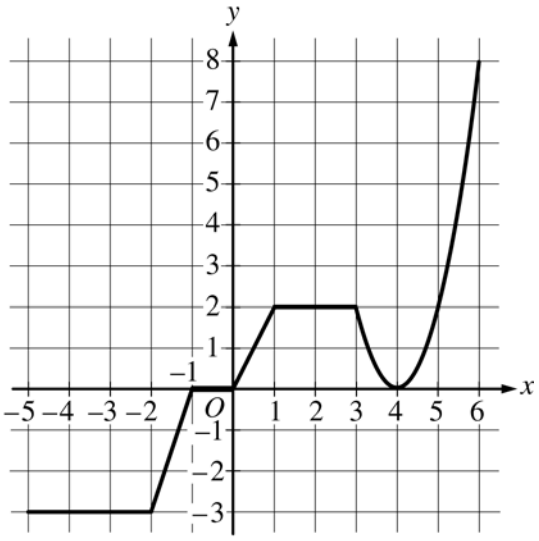

Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

   (a) If $f(1) = 3$, what is the value of $f(-5)$?

   (b) Evaluate $\int_1^6 g(x)\,dx$.

   (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

   (d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

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Question 3

(a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^1 g(x)\,dx$
$= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$
$= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx$
$= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$

$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

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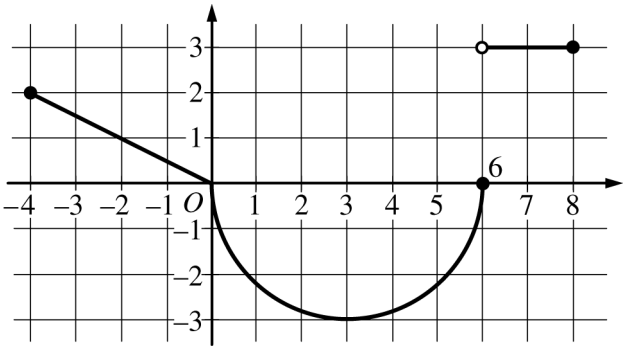

Graph of g

3. The function g is defined on the closed interval $[-4, 8]$. The graph of g consists of two linear pieces and a semicircle, as shown in the figure above. Let f be the function defined by $f(x) = 3x + \int_0^x g(t)\,dt$.

(a) Find $f(7)$ and $f'(7)$.

(b) Find the value of x in the closed interval $[-4, 3]$ at which f attains its maximum value. Justify your answer.

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Continue question 3 on page 15.

(c) For each of $\lim_{x \to 0^-} g'(x)$ and $\lim_{x \to 0^+} g'(x)$, find the value or state that it does not exist.

(d) Find $\lim_{x \to -2} \dfrac{f(x) + 7}{e^{3x+6} - 1}$.

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Question 3

(a) $f(7) = 3 \cdot 7 + \int_0^7 g(t)\,dt = 21 - \frac{9\pi}{2} + 3 = 24 - \frac{9\pi}{2}$ $f'(7) = 3 + g(7) = 3 + 3 = 6$	$2 : \begin{cases} 1 : f(7) \\ 1 : f'(7) \end{cases}$
(b) On the interval $-4 \le x \le 3,\ f'(x) = 3 + g(x)$. Because $f'(x) \ge 0$ for $-4 \le x \le 3,\ f$ is nondecreasing over the entire interval, and the maximum must occur when $x = 3$.	2 : answer with justification
(c) $\lim_{x \to 0^-} g'(x) = -\frac{1}{2}$ $\lim_{x \to 0^+} g'(x)$ does not exist.	$2 : \begin{cases} 1 : \text{left-hand limit} \\ 1 : \text{right-hand limit} \end{cases}$
(d) $\lim_{x \to -2} (f(x) + 7) = -6 + \int_0^{-2} g(t)\,dt + 7 = 0$ $\lim_{x \to -2} \left(e^{3x+6} - 1\right) = 0$ Using L'Hospital's Rule, $\lim_{x \to -2} \frac{f(x) + 7}{e^{3x+6} - 1} = \lim_{x \to -2} \frac{f'(x)}{3e^{3x+6}} = \frac{3 + g(-2)}{3} = \frac{3+1}{3} = \frac{4}{3}$.	$3 : \begin{cases} 1 : \text{limits equal } 0 \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$ Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

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t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

   (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

   (b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

   (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

   (d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

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Question 4

(a) $H'(6) \approx \dfrac{H(7) - H(5)}{7 - 5} = \dfrac{11 - 6}{2} = \dfrac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\dfrac{H(5) - H(3)}{5 - 3} = \dfrac{6 - 2}{2} = 2$

Because H is differentiable on $3 \le t \le 5$, H is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value c, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \dfrac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\dfrac{1}{10 - 2}\displaystyle\int_2^{10} H(t)\, dt$.

$$\frac{1}{8}\int_2^{10} H(t)\, dt \approx \frac{1}{8}\left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3\right)$$
$$= \frac{1}{8}(65.75) = \frac{263}{32}$$

The average height of the tree over the time interval $2 \le t \le 10$ is $\dfrac{263}{32}$ meters.

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1 + x)100 - 100x \cdot 1}{(1 + x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1 + x)^2} \cdot \frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{(1 + 1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\dfrac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \dfrac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0] if no chain rule

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4. Let g be the function that satisfies $g(0) = 0$ and whose derivative satisfies $g'(x) = 2|x|$.

(a) Find expressions for $g(x)$ and $g''(x)$.

(b) Find the x-coordinate, if any, of each point of inflection of the graph of $y = g(x)$. Explain your reasoning.

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Continue question 4 on page 17.

(c) Let $h(x) = \int_0^x \sqrt{1 + 4t^2}\, dt$. For $x \geq 0$, $h(x)$ is the length of the graph of g from $t = 0$ to $t = x$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(4)$.

(d) Find the value of $\int_{\pi/2}^{\pi} g'(x) \cos x\, dx$.

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Question 4

(a) $g(x) = \begin{cases} -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ x^2 & \text{for } x > 0 \end{cases}$

$g''(x) = \begin{cases} -2 & \text{for } x < 0 \\ 2 & \text{for } x > 0 \end{cases}$

$2 : \begin{cases} 1 : g(x) \\ 1 : g''(x) \end{cases}$

(b) The only point of inflection of the graph of $y = g(x)$ occurs at $x = 0$, where $g''(x)$ changes sign from negative to positive.

1 : answer with explanation

(c) $h(x) = \int_0^x \sqrt{1 + 4t^2}\, dt \Rightarrow h'(x) = \sqrt{1 + 4x^2}$

$h(0) = 0$

$h(2) \approx h(0) + 2 \cdot h'(0) = 0 + 2 \cdot 1 = 2$

$h(4) \approx h(2) + 2 \cdot h'(2) \approx 2 + 2\sqrt{17}$

$3 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation} \end{cases}$

(d) $g'(x) = 2x$ for $x > 0$

$u = 2x \qquad dv = \cos x\, dx$

$du = 2\, dx \qquad v = \sin x$

$$\int_{\pi/2}^{\pi} 2x \cos x\, dx = 2x \sin x \Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} 2 \sin x\, dx$$

$$= 2x \sin x \Big|_{\pi/2}^{\pi} + 2 \cos x \Big|_{\pi/2}^{\pi}$$

$$= 0 - \pi + (-2) - 0$$

$$= -\pi - 2$$

$3 : \begin{cases} 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

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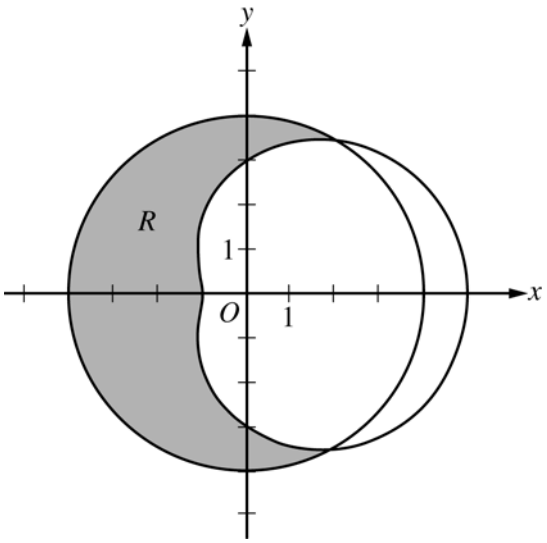

5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

   (a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

   (b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

   (c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

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Question 5

(a) $\text{Area} = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left(4^2 - (3+2\cos\theta)^2\right)d\theta$

$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

(b) $\frac{dr}{d\theta} = -2\sin\theta \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=\pi/2} = -2$

$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$

$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

— OR —

$y = r\sin\theta = (3+2\cos\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

$x = r\cos\theta = (3+2\cos\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

$3 : \begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \\ \quad \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -2\sin\theta\cdot\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta}$

$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3\cdot\frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$ radians per second

$3 : \begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$

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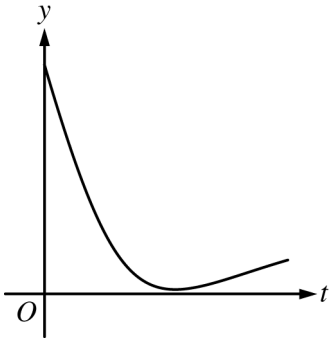
5. During a chemical reaction, the function $y = f(t)$ models the amount of a substance present, in grams, at time t seconds. At the start of the reaction $(t = 0)$, there are 10 grams of the substance present. The function $y = f(t)$ satisfies the differential equation $\frac{dy}{dt} = -0.02y^2$.

(a) Use the line tangent to the graph of $y = f(t)$ at $t = 0$ to approximate the amount of the substance remaining at time $t = 2$ seconds.

(b) Using the given differential equation, determine whether the graph of f could resemble the following graph. Give a reason for your answer.



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Continue question 5 on page 19.

(c) Find an expression for $y = f(t)$ by solving the differential equation $\frac{dy}{dt} = -0.02y^2$ with the initial condition $f(0) = 10$.

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(d) Determine whether the amount of the substance is changing at an increasing or a decreasing rate. Explain your reasoning.

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Question 5

(a) $y'(0) = -0.02\left(10^2\right) = -2$

An equation for the line tangent to the graph of $y = f(t)$ at $t = 0$ is $y = 10 - 2t$.

$y(2) \approx 10 - 2(2) = 6$ grams

$2 : \begin{cases} 1 : y'(0) \\ 1 : \text{approximation} \end{cases}$

(b) $\frac{dy}{dt} = -0.02y^2 \le 0,$ so the graph of f is nonincreasing.

The graph of f cannot resemble the given graph because the given graph is increasing on a portion of its domain.

1 : answer with reason

(c) $\int\left(-\frac{1}{y^2}\right)dy = \int 0.02\,dt$

$\frac{1}{y} = 0.02t + C$

$\frac{1}{10} = 0.02(0) + C \Rightarrow C = 0.1$

$\frac{1}{y} = 0.02t + 0.1 \Rightarrow y = \frac{1}{0.02t + 0.1} = \frac{50}{t + 5}$

Note: this solution is valid for $t > -5$.

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $2/4$ [1-1-0-0] if no constant of integration

Note: $0/4$ if no separation of variables

(d) $\frac{d^2y}{dt^2} = -0.04y\frac{dy}{dt}$

$= -0.04y\left(-0.02y^2\right)$

$= 0.0008y^3$

Because $y > 0$, $0.0008y^3 > 0$.
The amount of substance is changing at an increasing rate.

— OR —

From part (c), $f(t) = \frac{50}{t + 5},$ and from context, $t \ge 0$.

$f'(t) = \frac{-50}{(t + 5)^2}$ and $f''(t) = \frac{100}{(t + 5)^3} > 0$ for $t \ge 0$.

The amount of substance is changing at an increasing rate.

$2 : \begin{cases} 1 : \frac{d^2y}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

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6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

STOP

END OF EXAM

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Question 6

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

2 : { 1 : first four terms; 1 : general term }

(b) $\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)\left(3^{n+1}\right)}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \to \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$

$\left| \frac{x}{3} \right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

5 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : radius of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

— OR —

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left| \frac{x}{3} \right| < 1$.

$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

5 : { 1 : radius for $\ln(1+x)$ series; 1 : substitutes $\frac{x}{3}$; 1 : radius of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$

2 : { 1 : uses fifth-degree term as error bound; 1 : answer }

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x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
0	4	5	–1	$-\frac{15}{2}$	23
1	8	3	–2	$\frac{3}{2}$	$\frac{2}{5}$

6. Let f be a function having derivatives of all orders for all real numbers. Selected values of f and its first four derivatives are shown in the table above.

(a) Write the second-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

(b) Let g be a function such that $g(x) = f\left(x^3\right)$. Write the fifth-degree Taylor polynomial for g', the derivative of g, about $x = 0$.

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Continue question 6 on page 21.

(c) Write the third-degree Taylor polynomial for f about $x = 1$.

(d) It is known that $\left| f^{(4)}(x) \right| \leq 300$ for $0 \leq x \leq 1.125$. The third-degree Taylor polynomial for f about $x = 1$, found in part (c), is used to approximate $f(1.1)$. Use the Lagrange error bound along with the information about $f^{(4)}(x)$ to find an upper bound on the error of the approximation.

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Question 6

(a) The second-degree Taylor polynomial for f about $x = 0$ is $4 + 5x - \frac{1}{2}x^2$.

$f(0.2) \approx 4 + 5(0.2) - \frac{1}{2}(0.04) = 4.98$

3 : { 2 : second-degree Taylor polynomial; 1 : approximation }

(b) The fifth-degree Taylor polynomial for g' about $x = 0$ is $\frac{d}{dx}\left(4 + 5\left(x^3\right) - \frac{1}{2}\left(x^3\right)^2\right) = 15x^2 - 3x^5$.

2 : { 1 : substitution; 1 : answer }

(c) The third-degree Taylor polynomial for f about $x = 1$ is

$8 + 3(x-1) - \frac{2}{2}(x-1)^2 + \frac{3/2}{3!}(x-1)^3$

$= 8 + 3(x-1) - (x-1)^2 + \frac{1}{4}(x-1)^3$.

2 : { 1 : two terms; 1 : remaining terms }

(d) $\frac{\max\limits_{1\le x\le 1.1}\left|f^{(4)}(x)\right|}{4!}\cdot(1.1-1)^4 \le \frac{300}{4!}\cdot(1.1-1)^4 = \frac{300}{24}\cdot 0.1^4 = \frac{1}{800}$

An upper bound on the error of the approximation is $\frac{1}{800}$.

2 : { 1 : form of the error bound; 1 : answer }

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CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

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Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

2 : { 1 : integral; 1 : answer }

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

2 : { 1 : integral; 1 : answer }

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \le 8$. Therefore, the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore, the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

3 : { 1 : sets $E(t) - L(t) = 0$; 1 : answer; 1 : justification }

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

2 : { 1 : considers $E'(5)$ and $L'(5)$; 1 : answer with explanation }

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

t (hours)	2	5	9	11	12
$L(t)$ (cars per hour)	15	40	24	68	18

1. The rate at which cars enter a parking lot is modeled by $E(t) = 30 + 5(t - 2)(t - 5)e^{-0.2t}$. The rate at which cars leave the parking lot is modeled by the differentiable function L. Selected values of $L(t)$ are given in the table above. Both $E(t)$ and $L(t)$ are measured in cars per hour, and time t is measured in hours after 5 A.M. ($t = 0$). Both functions are defined for $0 \le t \le 12$.

   (a) What is the rate of change of $E(t)$ at time $t = 7$? Indicate units of measure.

   (b) How many cars enter the parking lot from time $t = 0$ to time $t = 12$? Give your answer to the nearest whole number.

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Continue question 1 on page 5.

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate $\int_2^{12} L(t)\,dt$. Using correct units, explain the meaning of $\int_2^{12} L(t)\,dt$ in the context of this problem.

(d) For $0 \le t < 6$, 5 dollars are collected from each car entering the parking lot. For $6 \le t \le 12$, 8 dollars are collected from each car entering the parking lot. How many dollars are collected from the cars entering the parking lot from time $t = 0$ to time $t = 12$? Give your answer to the nearest whole dollar.

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Question 1

t (hours)	2	5	9	11	12
$L(t)$ (cars per hour)	15	40	24	68	18

1. The rate at which cars enter a parking lot is modeled by $E(t) = 30 + 5(t-2)(t-5)e^{-0.2t}$. The rate at which cars leave the parking lot is modeled by the differentiable function L. Selected values of $L(t)$ are given in the table above. Both $E(t)$ and $L(t)$ are measured in cars per hour, and time t is measured in hours after 5 A.M. ($t = 0$). Both functions are defined for $0 \le t \le 12$.
   (a) What is the rate of change of $E(t)$ at time $t = 7$? Indicate units of measure.
   (b) How many cars enter the parking lot from time $t = 0$ to time $t = 12$? Give your answer to the nearest whole number.
   (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate $\int_2^{12} L(t)\,dt$. Using correct units, explain the meaning of $\int_2^{12} L(t)\,dt$ in the context of this problem.
   (d) For $0 \le t < 6$, 5 dollars are collected from each car entering the parking lot. For $6 \le t \le 12$, 8 dollars are collected from each car entering the parking lot. How many dollars are collected from the cars entering the parking lot from time $t = 0$ to time $t = 12$? Give your answer to the nearest whole dollar.

(a) $E'(7) = 6.164924$

The rate of change of $E(t)$ at time $t = 7$ is 6.165 (or 6.164) cars per hour per hour.

1 : answer with units

(b) $\int_0^{12} E(t)\,dt = 520.070489$

To the nearest whole number, 520 cars enter the parking lot from time $t = 0$ to time $t = 12$.

2 : { 1 : integrand; 1 : answer }

(c) $$\int_2^{12} L(t)\,dt \approx (5-2)\cdot\frac{L(2)+L(5)}{2} + (9-5)\cdot\frac{L(5)+L(9)}{2} + (11-9)\cdot\frac{L(9)+L(11)}{2} + (12-11)\cdot\frac{L(11)+L(12)}{2}$$
$$= 3\cdot\frac{15+40}{2} + 4\cdot\frac{40+24}{2} + 2\cdot\frac{24+68}{2} + 1\cdot\frac{68+18}{2}$$
$$= 345.5$$

$\int_2^{12} L(t)\,dt$ is the number of cars that leave the parking lot in the 10 hours between 7 A.M. ($t = 2$) and 5 P.M. ($t = 12$).

3 : { 1 : trapezoidal sum; 1 : approximation; 1 : explanation }

(d) $5\int_0^6 E(t)\,dt + 8\int_6^{12} E(t)\,dt = 3530.1396$

To the nearest dollar, 3530 dollars are collected from time $t = 0$ to time $t = 12$.

3 : { 1 : integrand; 1 : limits and constants; 1 : answer }

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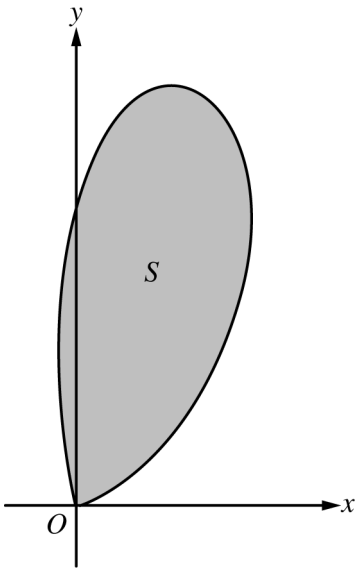

2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

   (a) Find the area of S.

   (b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

   (c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

   (d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

END OF PART A OF SECTION II

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Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$

The area of S is 3.534.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\tan\theta = \frac{y}{x} = m \implies \theta = \tan^{-1} m$

$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$

$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function applied to } m \text{ as limit of integration} \\ 1 : \text{equation} \end{cases}$

(d) As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis.

$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 \, d\theta = 3.324$

$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

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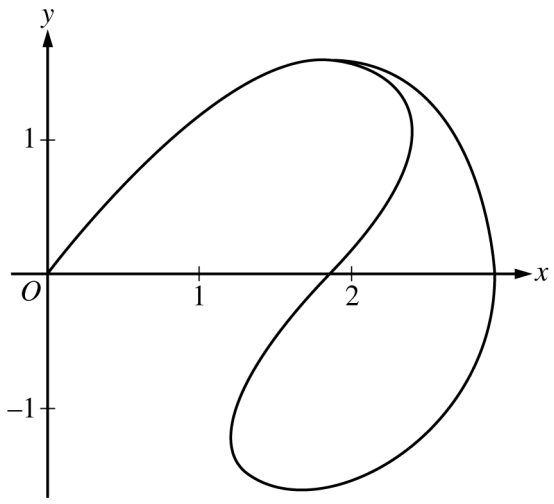

2. A laser is a device that produces a beam of light. A design, shown above, is etched onto a flat piece of metal using a moving laser. The position of the laser at time t seconds is represented by $(x(t), y(t))$ in the xy-plane. Both x and y are measured in centimeters, and t is measured in seconds. The laser starts at position $(0, 0)$ at time $t = 0$, and the design takes 3.1 seconds to complete. For $0 \le t \le 3.1$, $\frac{dx}{dt} = 3\cos\left(t^2\right)$ and $\frac{dy}{dt} = 4\cos(2.5t)$.

(a) Find the speed of the laser at time $t = 3$ seconds.

(b) Find the total distance traveled by the laser from time $t = 1$ to time $t = 3$ seconds.

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Continue question 2 on page 7.

(c) The laser is farthest to the right at time $t = 1.253$ seconds. Find the x-coordinate of the laser's rightmost position.

(d) What is the difference between the y-coordinates of the laser's highest position and lowest position for $0 \le t \le 3.1$? Justify your answer.

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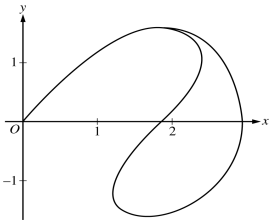
Question 2



2. A laser is a device that produces a beam of light. A design, shown above, is etched onto a flat piece of metal using a moving laser. The position of the laser at time t seconds is represented by $(x(t), y(t))$ in the xy-plane. Both x and y are measured in centimeters, and t is measured in seconds. The laser starts at position $(0, 0)$ at time $t = 0$, and the design takes 3.1 seconds to complete. For $0 \le t \le 3.1$, $\frac{dx}{dt} = 3\cos\left(t^2\right)$ and $\frac{dy}{dt} = 4\cos(2.5t)$.

(a) Find the speed of the laser at time $t = 3$ seconds.

(b) Find the total distance traveled by the laser from time $t = 1$ to time $t = 3$ seconds.

(c) The laser is farthest to the right at time $t = 1.253$ seconds. Find the x-coordinate of the laser's rightmost position.

(d) What is the difference between the y-coordinates of the laser's highest position and lowest position for $0 \le t \le 3.1$? Justify your answer.

(a) $\sqrt{(x'(3))^2 + (y'(3))^2} = 3.064951$

The speed of the laser at time $t = 3$ seconds is 3.065 (or 3.064) centimeters per second.

1 : answer

(b) $\int_1^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 7.090427$

The total distance traveled by the laser from time $t = 1$ to time $t = 3$ seconds is 7.090 centimeters.

2 : { 1 : integral ; 1 : answer }

(c) $x(1.253) = 0 + \int_0^{1.253} \frac{dx}{dt}\, dt = 2.932354$

The x-coordinate of the laser's rightmost position is 2.932.

2 : { 1 : integral ; 1 : answer }

(d) $\frac{dy}{dt} = 0 \Rightarrow t = t_1 = 0.628319$ or $t = t_2 = 1.884956$

$y(0) = 0$

$y(t_1) = 0 + \int_0^{t_1} \frac{dy}{dt}\, dt = 1.6$

$y(t_2) = 0 + \int_0^{t_2} \frac{dy}{dt}\, dt = -1.6$

$y(3.1) = 0 + \int_0^{3.1} \frac{dy}{dt}\, dt = 1.591358$

The difference between the y-coordinates of the laser's highest position and lowest position is $y(t_1) - y(t_2) = 3.2$.

4 : { 1 : sets $\frac{dy}{dt} = 0$; 1 : critical points ; 1 : integrand ; 1 : answer with justification }

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CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

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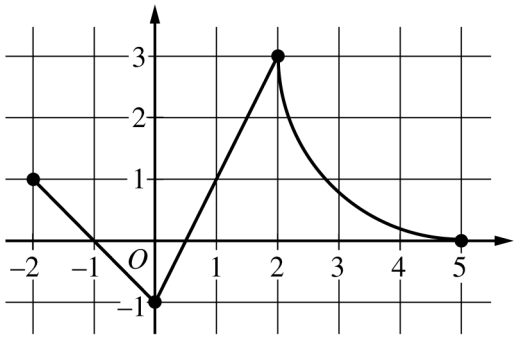

Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

   (a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

   (b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

   (c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

   (d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

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Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

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$$f(x) = \begin{cases} \sqrt{9 - x^2} & \text{for } -3 \le x \le 0 \\ -x + 3\cos\left(\dfrac{\pi x}{2}\right) & \text{for } 0 < x \le 4 \end{cases}$$

3. Let f be the function defined above.

(a) Find the average rate of change of f on the interval $-3 \le x \le 4$.

(b) Write an equation for the line tangent to the graph of f at $x = 3$.

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Continue question 3 on page 15.

(c) Find the average value of f on the interval $-3 \le x \le 4$.

(d) Must there be a value of x at which $f(x)$ attains an absolute maximum on the closed interval $-3 \le x \le 4$? Justify your answer.

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Question 3

$$f(x) = \begin{cases} \sqrt{9 - x^2} & \text{for } -3 \le x \le 0 \\ -x + 3\cos\left(\frac{\pi x}{2}\right) & \text{for } 0 < x \le 4 \end{cases}$$

3. Let f be the function defined above.

(a) Find the average rate of change of f on the interval $-3 \le x \le 4$.
(b) Write an equation for the line tangent to the graph of f at $x = 3$.
(c) Find the average value of f on the interval $-3 \le x \le 4$.
(d) Must there be a value of x at which $f(x)$ attains an absolute maximum on the closed interval $-3 \le x \le 4$? Justify your answer.

(a) Average rate of change $= \dfrac{f(4) - f(-3)}{4 - (-3)} = \dfrac{-1 - 0}{7} = -\dfrac{1}{7}$

1 : answer

(b) $f(3) = -3 + 3\cos\left(\dfrac{3\pi}{2}\right) = -3$

For $0 < x < 4$, $f'(x) = -1 + \left(-3\sin\left(\dfrac{\pi x}{2}\right)\right) \cdot \dfrac{\pi}{2}$

$f'(3) = -1 + \left(-3\sin\left(\dfrac{3\pi}{2}\right)\right) \cdot \dfrac{\pi}{2} = -1 + \dfrac{3\pi}{2}$

An equation for the tangent line is $y = -3 + \left(-1 + \dfrac{3\pi}{2}\right)(x - 3)$.

2 : { 1 : $f'(3)$; 1 : equation }

(c) The average value of f on the interval $-3 \le x \le 4$ is $\dfrac{1}{4 - (-3)} \displaystyle\int_{-3}^{4} f(x)\, dx$.

$\displaystyle\int_{-3}^{4} f(x)\, dx = \int_{-3}^{0} f(x)\, dx + \int_{0}^{4} f(x)\, dx$

$\displaystyle\int_{-3}^{0} f(x)\, dx = \int_{-3}^{0} \sqrt{9 - x^2}\, dx = \frac{9\pi}{4}$

$\displaystyle\int_{0}^{4} f(x)\, dx = \int_{0}^{4} \left(-x + 3\cos\left(\frac{\pi x}{2}\right)\right) dx = \left[-\frac{1}{2}x^2 + \frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^4 = -8$

$\dfrac{1}{4 - (-3)} \displaystyle\int_{-3}^{4} f(x)\, dx = \frac{1}{7}\left(\frac{9\pi}{4} - 8\right)$

4 : { 1 : integrals of f over $-3 \le x \le 0$ and $0 \le x \le 4$; 1 : value of $\displaystyle\int_{-3}^{0} \sqrt{9 - x^2}\, dx$; 1 : antiderivative of $-x + 3\cos\left(\dfrac{\pi x}{2}\right)$; 1 : answer }

(d) $\lim\limits_{x \to 0^-} f(x) = f(0) = 3$ and $\lim\limits_{x \to 0^+} f(x) = 3$, so f is continuous at $x = 0$.

Because f is continuous on $[-3, 4]$, the Extreme Value Theorem guarantees that f attains an absolute maximum on $[-3, 4]$.

2 : { 1 : continuity at $x = 0$; 1 : answer with justification }

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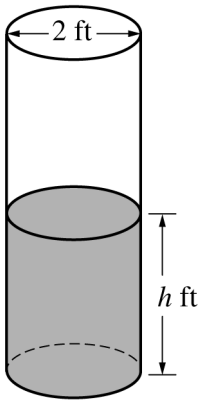

4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

   (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

   (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

   (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

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Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second

$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$

Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$

$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{1}{10}t + C$

$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

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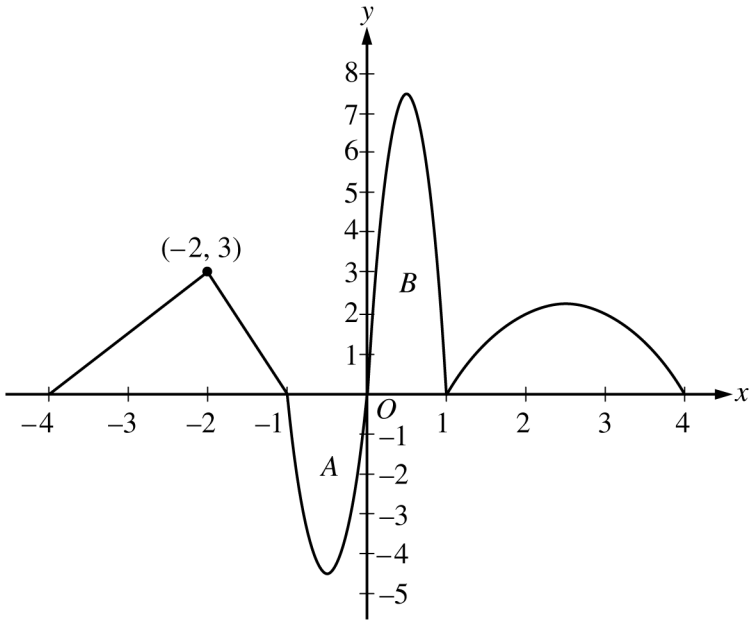

Graph of f

4. The continuous function f is defined for $-4 \le x \le 4$. The graph of f, shown above, consists of two line segments and portions of three parabolas. The graph has horizontal tangents at $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = \frac{5}{2}$. It is known that $f(x) = -x^2 + 5x - 4$ for $1 \le x \le 4$. The areas of regions A and B bounded by the graph of f and the x-axis are 3 and 5, respectively. Let g be the function defined by $g(x) = \int_{-4}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g(4)$.

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Continue question 4 on page 17.

(b) Find the absolute minimum value of g on the closed interval $[-4, 4]$. Justify your answer.

(c) Find all intervals on which the graph of g is concave down. Give a reason for your answer.

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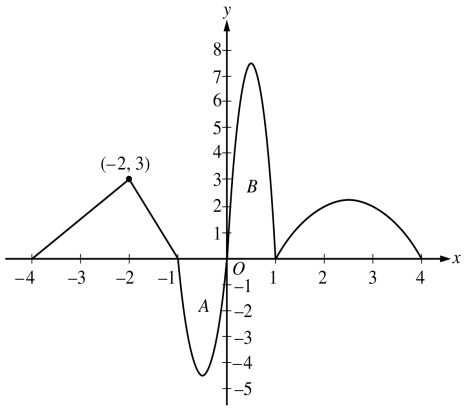
Question 4



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   (a) Find $g(0)$ and $g(4)$.
   (b) Find the absolute minimum value of g on the closed interval $[-4, 4]$. Justify your answer.
   (c) Find all intervals on which the graph of g is concave down. Give a reason for your answer.

(a) $g(0) = \int_{-4}^{0} f(t)\,dt = \frac{9}{2} - 3 = \frac{3}{2}$

$g(4) = \int_{-4}^{4} f(t)\,dt$

$= \int_{-4}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{4} f(t)\,dt$

$= \frac{3}{2} + 5 + \int_{1}^{4} \left(-t^2 + 5t - 4\right) dt$

$= \frac{3}{2} + 5 + \left[-\frac{1}{3}t^3 + \frac{5}{2}t^2 - 4t\right]_{1}^{4}$

$= \frac{3}{2} + 5 + \left[\left(-\frac{1}{3}\cdot 4^3 + \frac{5}{2}\cdot 4^2 - 4\cdot 4\right) - \left(-\frac{1}{3}\cdot 1^3 + \frac{5}{2}\cdot 1^2 - 4\cdot 1\right)\right]$

$= \frac{3}{2} + 5 + \left(\frac{8}{3} - \left(-\frac{11}{6}\right)\right) = 11$

4 : { 1 : $g(0)$; 1 : integral of f over $1 \le t \le 4$; 1 : antiderivative; 1 : $g(4)$

(b) $g'(x) = f(x)$ is negative for $-1 < x < 0$, and nonnegative elsewhere. Thus, the absolute minimum value of g on $[-4, 4]$ can only occur at $x = -4$ or $x = 0$.

$g(-4) = 0 \qquad g(0) = \frac{3}{2}$

The absolute minimum value of g on $[-4, 4]$ is $g(-4) = 0$.

3 : { 1 : $g'(x) = f(x)$; 1 : identifies $x = -4$ and $x = 0$ as candidates; 1 : answer with justification

(c) The graph of g is concave down on the intervals $-2 < x < -\frac{1}{2}$, $\frac{1}{2} < x < 1$, and $\frac{5}{2} < x < 4$ because $g'(x) = f(x)$ is decreasing on these intervals.

2 : { 1 : intervals; 1 : reason

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5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

   (a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

   (b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

   (c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

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Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2-2x+k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

3 : { 1 : denominator of $f'(x)$; 1 : $f'(x)$; 1 : answer }

(b) $\dfrac{1}{x^2-2x-8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$\displaystyle\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}\right) dx$

$= \left[\dfrac{1}{6}\ln|x-4| - \dfrac{1}{6}\ln|x+2|\right]_{x=0}^{x=1}$

$= \left(\dfrac{1}{6}\ln 3 - \dfrac{1}{6}\ln 3\right) - \left(\dfrac{1}{6}\ln 4 - \dfrac{1}{6}\ln 2\right) = -\dfrac{1}{6}\ln 2$

3 : { 1 : partial fraction decomposition; 1 : antiderivatives; 1 : answer }

(c) $\displaystyle\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$

$\displaystyle= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$

$\displaystyle= \lim_{b\to 1^-} \left(-\frac{1}{x-1}\Big|_{x=0}^{x=b}\right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1}\Big|_{x=b}^{x=2}\right)$

$\displaystyle= \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1\right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1}\right)$

Because $\displaystyle\lim_{b\to 1^-} \left(-\frac{1}{b-1}\right)$ does not exist, the integral diverges.

3 : { 1 : improper integral; 1 : antiderivative; 1 : answer with reason }

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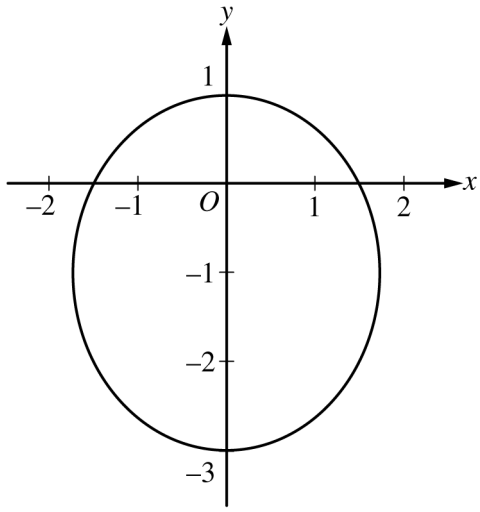

5. The graph of the curve C, given by $4x^2 + 3y^2 + 6y = 9$, is shown in the figure above.

(a) Show that $\frac{dy}{dx} = \frac{-4x}{3(y+1)}$.

(b) Using the information from part (a), find $\frac{d^2y}{dx^2}$ in terms of x and y.

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(c) In polar coordinates, the curve C is given by $r = \dfrac{3}{2 + \sin\theta}$ for $0 \le \theta \le 2\pi$. Find $\dfrac{dr}{d\theta}$.

As θ increases, on what intervals is the distance between the origin and the point (r, θ) increasing?

Give a reason for your answer.

(d) Let S be the region inside curve C, as defined in part (c), but outside the curve $r = 2$. Write, but do not evaluate, an integral expression for the area of S.

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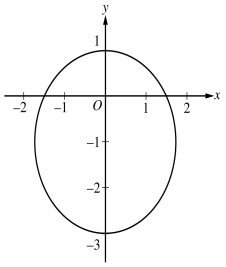
Question 5



5. The graph of the curve C, given by $4x^2 + 3y^2 + 6y = 9$, is shown in the figure above.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{-4x}{3(y+1)}$.

   (b) Using the information from part (a), find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

   (c) In polar coordinates, the curve C is given by $r = \dfrac{3}{2 + \sin\theta}$ for $0 \le \theta \le 2\pi$. Find $\dfrac{dr}{d\theta}$.

   As θ increases, on what intervals is the distance between the origin and the point (r, θ) increasing?

   Give a reason for your answer.

   (d) Let S be the region inside curve C, as defined in part (c), but outside the curve $r = 2$. Write, but do not evaluate, an integral expression for the area of S.

(a) $\dfrac{d}{dx}\left(4x^2 + 3y^2 + 6y\right) = \dfrac{d}{dx}(9)$

$\Rightarrow 8x + 6y\dfrac{dy}{dx} + 6\dfrac{dy}{dx} = 0$

$\Rightarrow (6y + 6)\dfrac{dy}{dx} = -8x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-8x}{6y + 6} = \dfrac{-4x}{3(y+1)}$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{verification} \end{cases}$

(b) $\dfrac{d^2y}{dx^2} = \dfrac{-4 \cdot 3(y+1) - (-4x) \cdot 3\dfrac{dy}{dx}}{(3(y+1))^2} = \dfrac{-12(y+1) - \dfrac{16x^2}{y+1}}{(3(y+1))^2}$

$2 : \begin{cases} 1 : \text{form of quotient rule} \\ 1 : \dfrac{d^2y}{dx^2} \end{cases}$

(c) $r = \dfrac{3}{2 + \sin\theta} > 0$ for $0 \le \theta \le 2\pi$.

$\dfrac{dr}{d\theta} = \dfrac{-3}{(2 + \sin\theta)^2} \cdot \cos\theta$

$(2 + \sin\theta)^2 > 0$ for all θ.

For $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$, $\cos\theta < 0$, so $\dfrac{dr}{d\theta} > 0$.

The distance between the origin and the point (r, θ) is increasing for $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.

$2 : \begin{cases} 1 : \dfrac{dr}{d\theta} \\ 1 : \text{answer with reason} \end{cases}$

(d) $2 = \dfrac{3}{2 + \sin\theta} \Rightarrow 2 + \sin\theta = \dfrac{3}{2} \Rightarrow \sin\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{7\pi}{6}$ or $\theta = \dfrac{11\pi}{6}$

Area $= \dfrac{1}{2}\displaystyle\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left(\left(\frac{3}{2 + \sin\theta}\right)^2 - 2^2\right) d\theta$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{form of integrand} \\ 1 : \text{integrand} \end{cases}$

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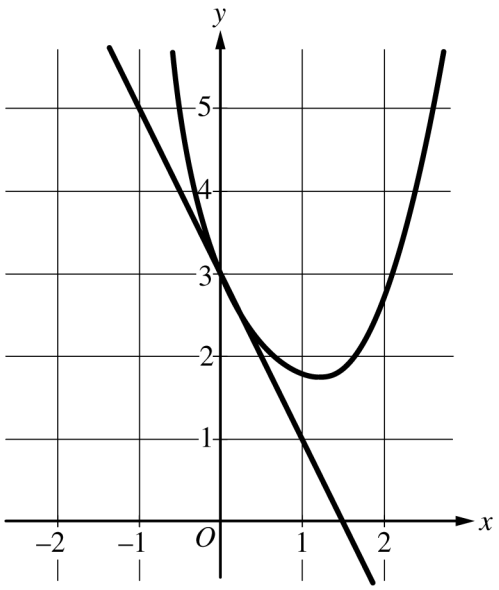

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

   (a) Write the third-degree Taylor polynomial for f about $x = 0$.

   (b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

   (c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

   (d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

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Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} &3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ &\quad = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ &\quad = 3 + x + x^2. \end{aligned}$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $h(1) = \int_0^1 f(t)\,dt$

$$\begin{aligned} &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$

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6. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{5^n \cdot n^p}$, where p is a constant and $p > 0$.

(a) For $p = 3$ and $x = 8$, does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

(b) For $p = 1$ and $x = 8$, does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

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Continue question 6 on page 21.

(c) When $x = -2$, for what values of p does the series converge? Explain your reasoning.

(d) When $p = 1$ and $x = 3.1$, the series converges to a value S. Use the first two terms of the series to approximate S. Use the alternating series error bound to show that this approximation differs from S by less than $\frac{1}{300{,}000}$.

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AP® CALCULUS BC
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Question 6

6. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{5^n \cdot n^p}$, where p is a constant and $p > 0$.

(a) For $p = 3$ and $x = 8$, does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

(b) For $p = 1$ and $x = 8$, does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

(c) When $x = -2$, for what values of p does the series converge? Explain your reasoning.

(d) When $p = 1$ and $x = 3.1$, the series converges to a value S. Use the first two terms of the series to approximate S. Use the alternating series error bound to show that this approximation differs from S by less than $\frac{1}{300{,}000}$.

(a) The series with $p = 3$ and $x = 8$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^n}{5^n \cdot n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 > 1$, which converges.

Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^n}{5^n \cdot n^3}$ converges absolutely.

$2 : \begin{cases} 1 : \text{considers } \sum_{n=1}^{\infty} \frac{1}{n^3} \\ 1 : \text{converges absolutely with explanation} \end{cases}$

(b) The series with $p = 1$ and $x = 8$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^n}{5^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.

Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^n}{5^n \cdot n}$ converges conditionally.

$2 : \begin{cases} 1 : \text{considers } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \\ 1 : \text{converges conditionally with explanation} \end{cases}$

(c) The series with $x = -2$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5)^n}{5^n \cdot n^p} = \sum_{n=1}^{\infty} \frac{-1}{n^p} = -\sum_{n=1}^{\infty} \frac{1}{n^p}$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series, which converges if and only if $p > 1$.

Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5)^n}{5^n \cdot n^p}$ converges for $p > 1$.

$2 : \begin{cases} 1 : \text{considers } \sum_{n=1}^{\infty} \frac{-1}{n^p} \\ 1 : \text{answer with explanation} \end{cases}$

(d) The series with $p = 1$ and $x = 3.1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n \cdot n}(3.1-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.1)^n}{5^n \cdot n}$.

Using two terms, $S \approx \frac{0.1}{5} - \frac{(0.1)^2}{5^2 \cdot 2} = \frac{99}{5000}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.1)^n}{5^n \cdot n}$ is an alternating series with terms that decrease in magnitude to 0.

By the alternating series error bound, the approximation $S \approx \frac{99}{5000}$ has absolute error bounded by the magnitude of the third term, $\frac{1}{5^3 \cdot 10^3 \ \ 3}$.

Therefore, $\left| S - \frac{99}{5000} \right| < \left| \frac{1}{5^3 \cdot 10^3 \cdot 3} \right| = \frac{1}{375{,}000} < \frac{1}{300{,}000}$.

$3 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{uses third term of series} \\ 1 : \text{error bound} \end{cases}$

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r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

   (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

   (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

   (d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

r (centimeters)	0	1	2	2.5	4
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(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

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Part A (AB or BC): Graphing calculator required
Question 1 — 9 points

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

	Model Solution	Scoring	
(a)	Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
	$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = 8$	Estimate	**1 point**
	At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**
(b)	The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.		
	$2\pi\int_0^4 r f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
	$= 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5)$ $= 269\pi = 845.088$	Approximation	**1 point**
(c)	Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.		
	$\frac{d}{dr}(r f(r)) = f(r) + r f'(r)$	Product rule expression for $\frac{d}{dr}(r f(r))$	**1 point**
	Because f is nonnegative and increasing, $\frac{d}{dr}(r f(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $r f(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r f(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

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(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \frac{1}{4-1}\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\frac{1}{4-1}\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

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Part A (AB or BC): Graphing calculator required
Question 1 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

	Model Solution	Scoring	
(a)	Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
	$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = 8$	Estimate	**1 point**
	At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**

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Scoring notes:

- To earn the first point, the response must provide both a difference and a quotient and must explicitly use values of f from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25,$ density of bacteria (population) is increasing or changing, at a rate of $8,$ and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25)$.
- To earn the second point, the interpretation must be consistent with the presented nonzero value for $f'(2.25)$. In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25)$." The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of -8 …" even for a presented $f'(2.25) = -8$.
- The units ($\text{mg/cm}^2\text{/cm}$) may be attached to the estimate of $f'(2.25)$ and if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

Total for part (a) 2 points

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(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0)+2\cdot f(2)\cdot(2-1)$ $+2.5\cdot f(2.5)\cdot(2.5-2)+4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
$=2\pi(1\cdot 2\cdot 1+2\cdot 6\cdot 1+2.5\cdot 10\cdot 0.5+4\cdot 18\cdot 1.5)$ $=269\pi=845.088$	Approximation	**1 point**

Scoring notes:

- The presence or absence of 2π has no bearing on earning the first point.
- The first point is earned for a sum of 4 products with at most one error in any single value among the four products. Multiplication by 1 in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0)+2\cdot f(2)\cdot(2-1)+2.5\cdot f(2.5)\cdot(2.5-2)+4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation (91π) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation (80π) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

Total for part (b) 2 points

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(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$\frac{d}{dr}(rf(r)) = f(r) + rf'(r)$	Product rule expression for $\frac{d}{dr}(rf(r))$	**1 point**
Because f is nonnegative and increasing, $\frac{d}{dr}(rf(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $rf(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 rf(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

Scoring notes:

- To earn the second point, a response must explain that $rf(r)$ is increasing, and therefore the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 rf(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

Total for part (c) 2 points

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(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \frac{1}{4-1}\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\frac{1}{4-1}\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral, with or without $\frac{1}{4-1}$ or $\frac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value 9.875795 is not required to earn the second point. This point can be earned by the average value setup: $\frac{1}{3}\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of -13.955 is using degree mode on their calculator. This response would not earn the second point, but could earn the third point for an answer of $k = 2.5$ (or 2.499).

Total for part (d) 3 points

Total for question 1 9 points

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2. For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

   (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

   (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

   (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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Part A (BC): Graphing calculator required
Question 2 — 9 points

For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right)\right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

	Model Solution	Scoring	
(a)	Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.		
	$\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271488$ At time $t = 1.2$, the speed of the particle is 1.271.	Speed	**1 point**
	$\langle x''(1.2), y''(1.2)\rangle = \langle 6.246630, 0.405125\rangle$ At time $t = 1.2$, the acceleration vector of the particle is $\langle 6.247$ (or $6.246), 0.405\rangle$.	Acceleration vector	**1 point**
(b)	Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.		
	$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009817$	Integrand	**1 point**
	The total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$ is 1.010 (or 1.009).	Answer	**1 point**
(c)	Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.		
	$x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$	Sets $x'(t) = 0$	**1 point**
	Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest to the left at time $t = 1$.	Explains leftmost position at $t = 1$	**1 point**
	$x(1) = -2 + \int_0^1 x'(t)\, dt = -2.603511$ $y(1) = 5 + \int_0^1 y'(t)\, dt = 5.410486$	One coordinate of leftmost position	**1 point**
	The particle is farthest to the left at point $(-2.604$ (or $-2.603), 5.410)$.	Leftmost position	**1 point**
	Because $x'(t) > 0$ for $t > 1$, the particle moves right for $t > 1$. Also, $x(2) = -2 + \int_0^2 x'(t)\, dt > -2 = x(0)$, so the particle's motion extends to the right of its initial position after time $t = 1$. Therefore, there is no point at which the particle is farthest to the right.	Explanation	**1 point**

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Part A (BC): Graphing calculator required
Question 2 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right)\right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

	Model Solution	Scoring	
(a)	Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.		
	$\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271488$ At time $t = 1.2$, the speed of the particle is 1.271.	Speed	**1 point**
	$\langle x''(1.2), y''(1.2)\rangle = \langle 6.246630, 0.405125\rangle$ At time $t = 1.2$, the acceleration vector of the particle is $\langle 6.247$ (or $6.246), 0.405\rangle$.	Acceleration vector	**1 point**

Scoring notes:

- Unsupported answers do not earn any points in this part.
- The acceleration vector may be presented with other symbols, for example $(\ ,\ )$ or $[\ ,\ ]$, or the coordinates may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, speed $= 0.844$ and $y''(1.2) = 0.023$ (or 0.022).

Total for part (a) 2 points

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(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 1.2$.

$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009817$	Integrand	**1 point**
The total distance traveled by the particle over the time interval $0 \le t \le 1.2$ is 1.010 (or 1.009).	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting the integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with any limits. A definite integral with incorrect limits is not eligible for the second point.
- Once earned, the first point cannot be lost. Even in the presence of subsequent copy errors, the correct answer will earn the second point.
- If the first point is not earned because of a copy error, the second point is still earned for a correct answer.
- Unsupported answers will not earn either point.
- Degree mode: distance $= 0.677$ (or 0.676) (See degree mode statement in part (a).)

Total for part (b) 2 points

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(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

$x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$	Sets $x'(t) = 0$	**1 point**
Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest to the left at time $t = 1$.	Explains leftmost position at $t = 1$	**1 point**
$x(1) = -2 + \int_0^1 x'(t)\,dt = -2.603511$ $y(1) = 5 + \int_0^1 y'(t)\,dt = 5.410486$	One coordinate of leftmost position	**1 point**
The particle is farthest to the left at point $(-2.604 \text{ (or } -2.603), 5.410)$.	Leftmost position	**1 point**
Because $x'(t) > 0$ for $t > 1$, the particle moves right for $t > 1$. Also, $x(2) = -2 + \int_0^2 x'(t)\,dt > -2 = x(0)$, so the particle's motion extends to the right of its initial position after time $t = 1$. Therefore, there is no point at which the particle is farthest to the right.	Explanation	**1 point**

Scoring notes:

- The second point is earned for presenting a valid reason why the particle is at its leftmost position at time $t = 1$. For example, a response could present the argument shown in the model solution, or it could indicate that the only critical point of $x(t)$ occurs at $t = 1$ and $x'(t)$ changes from negative to positive at this time.
- Unsupported positions $x(1)$ and/or $y(1)$ do not earn the third (or fourth) point(s).
- Writing $x(1) = \int_0^1 x'(t) - 2 = -2.603511$ does not earn the third (or fourth) point, because the missing dt makes this statement unclear or false. However, $x(1) = -2 + \int_0^1 x'(t) = -2.603511$ does earn the third point, because it is not ambiguous. Similarly for $y(1)$.
- For the fourth point, the coordinates of the leftmost point do not have to be written as an ordered pair as long as they are labeled as the x- and y-coordinates.
- To earn the last point, a response must verify that the particle moves to the right of its initial position (as well as moves to the right for all $t > 1$). Note that there are several ways to demonstrate this.
- Degree mode: y-coordinate $= 5.008 \text{ (or } 5.007)$ (See degree mode statement in part (a).)

Total for part (c) 5 points

Total for question 2 9 points

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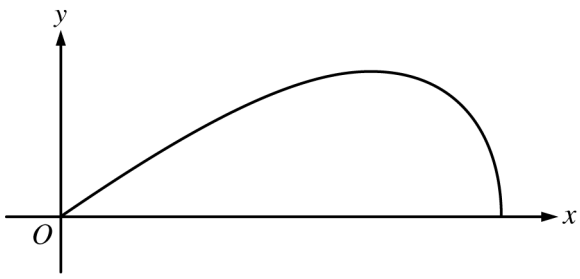

3. A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

   (a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

   (b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

   (c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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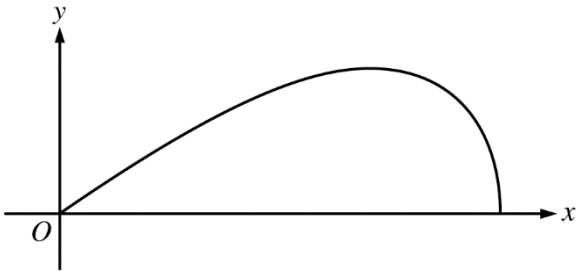
Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

	Model Solution	Scoring	
(a)	Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.		
	$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$ Area $= \int_0^2 6x\sqrt{4 - x^2}\, dx$	Integrand	**1 point**
	Let $u = 4 - x^2$. $du = -2x\, dx \Rightarrow -\frac{1}{2}\, du = x\, dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\, du = -3\int_4^0 u^{1/2}\, du = 3\int_0^4 u^{1/2}\, du$ $= 2u^{3/2}\Big\|_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
	The area of the region is 16 square inches.	Answer	**1 point**

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(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\frac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Volume $= \int_0^2 \pi\left(cx\sqrt{4 - x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2\left(4 - x^2\right) dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2 \int_0^2 \left(4x^2 - x^4\right) dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \Big\vert_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2 \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

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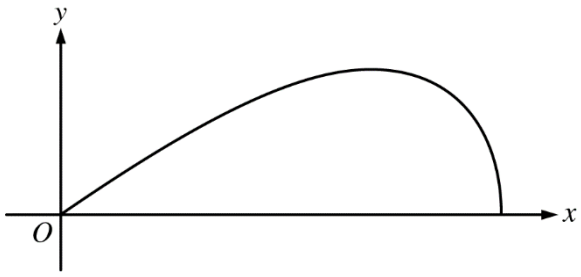
Part B (AB or BC): Graphing calculator not allowed
Question 3 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

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Model Solution	Scoring

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$ Area $= \int_0^2 6x\sqrt{4 - x^2}\, dx$	Integrand	**1 point**
Let $u = 4 - x^2$. $du = -2x\, dx \Rightarrow -\frac{1}{2}\, du = x\, dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\, du = -3\int_4^0 u^{1/2}\, du = 3\int_0^4 u^{1/2}\, du$ $= 2u^{3/2}\Big\|_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral), but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

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(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\frac{dy}{dx} = 0$ **1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer **1 point**

Scoring notes:

- The first point is earned for setting $\frac{dy}{dx} = 0$, $\frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0$, or $c\left(4 - 2x^2\right) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

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(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2\,dx = \pi c^2\int_0^2 x^2\left(4-x^2\right)\,dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2\int_0^2\left(4x^2-x^4\right)\,dx = \pi c^2\left(\frac{4}{3}x^3-\frac{1}{5}x^5\Big\|_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2\left(cx\sqrt{4-x^2}\right)^2\,dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

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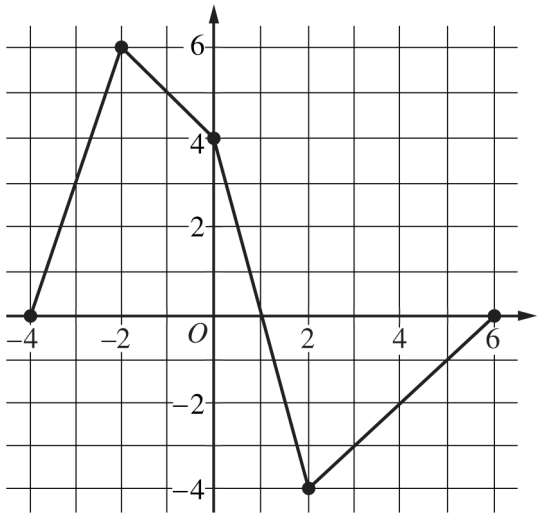

Graph of f

4. Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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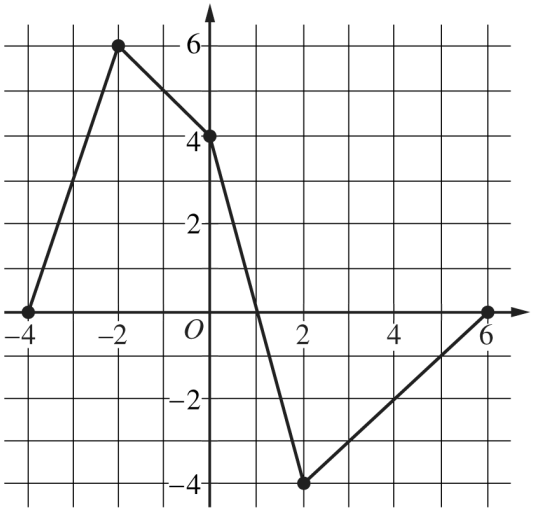
NO CALCULATOR ALLOWED



Graph of f

4. Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

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NO CALCULATOR ALLOWED

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

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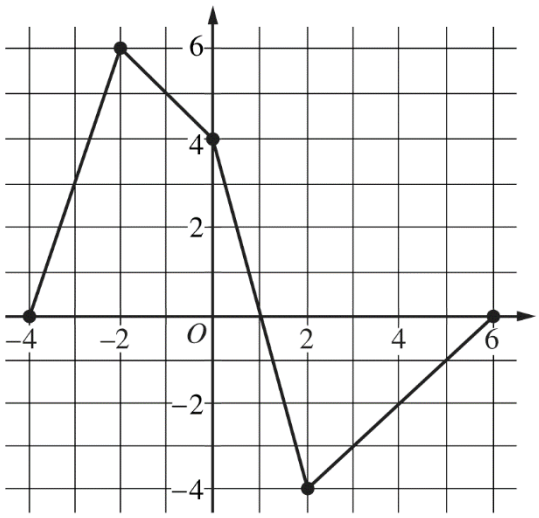
Part B (AB or BC): Graphing calculator not allowed

Question 4 — 9 points



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

	Model Solution	Scoring	
	$G'(x) = f(x)$ in any part of the response	$G'(x) = f(x)$	**1 point**
(a)	On what open intervals is the graph of G concave up? Give a reason for your answer.		
	$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**
(b)	Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.		
	$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
	Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
	$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

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(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

$\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ Because G is continuous for $-4 \le x \le 6$, $\lim_{x \to 2} G(x) = \int_0^2 f(t)\, dt = 0$. Therefore, the limit $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\frac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{G'(x)}{2x - 2}$ $= \lim_{x \to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$	Answer with justification	**1 point**

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\, dt = 0$ and $G(-4) = \int_0^{-4} f(t)\, dt = -16$ Average rate of change $= \frac{G(2) - G(-4)}{2 - (-4)} = \frac{0 - (-16)}{6} = \frac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \frac{8}{3}$.	Answer with justification	**1 point**

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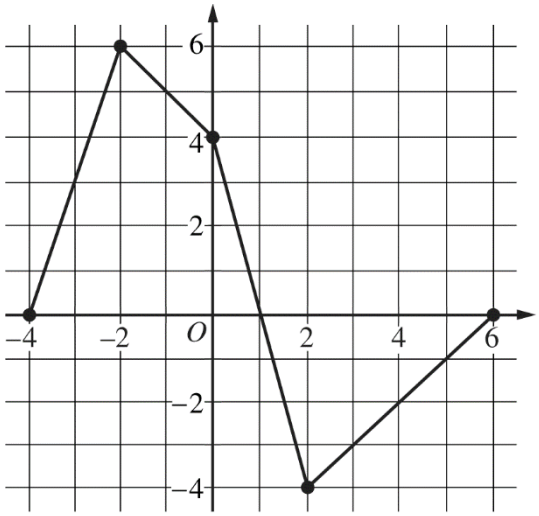
Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

Model Solution	Scoring	
$G'(x) = f(x)$ in any part of the response	$G'(x) = f(x)$	**1 point**

Scoring notes:

- This "global point" can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

Total 1 point

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(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**

Scoring notes:

- Intervals may also include one or both endpoints.

Total for part (a) 1 point

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for the correct application of the product rule in terms of x or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point, a response must have earned the first two points. Simplification of the numerical value is not required to earn the third point.

Total for part (b) 3 points

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(c) Find $\lim_{x\to 2}\dfrac{G(x)}{x^2-2x}$.

$\lim_{x\to 2}\left(x^2-2x\right)=0$ Because G is continuous for $-4\le x\le 6$, $\lim_{x\to 2}G(x)=\int_0^2 f(t)\,dt=0$. Therefore, the limit $\lim_{x\to 2}\dfrac{G(x)}{x^2-2x}$ is an indeterminate form of type $\dfrac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x\to 2}\dfrac{G(x)}{x^2-2x}=\lim_{x\to 2}\dfrac{G'(x)}{2x-2}$ $=\lim_{x\to 2}\dfrac{f(x)}{2x-2}=\dfrac{f(2)}{2}=\dfrac{-4}{2}=-2$	Answer with justification	**1 point**

Scoring notes:

- To earn the first point, the response must show $\lim_{x\to 2}\left(x^2-2x\right)=0$ and $\lim_{x\to 2}G(x)=0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x-2$. The ratio may be shown as evaluations of the derivatives at $x=2$, such as $\dfrac{G'(2)}{2}$.
- To earn the second point, the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x\to 2}\dfrac{G'(x)}{2x-2}$ or $\lim_{x\to 2}\dfrac{f(x)}{2x-2}$.
- With any linkage errors (such as $\dfrac{G'(x)}{2x-2}=\dfrac{f(2)}{2}$), the response does not earn the second point.

Total for part (c) 2 points

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(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$ Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$.	Answer with justification	**1 point**

Scoring notes:

- To earn the first point, a response must present at least a difference and a quotient and a correct evaluation. For example, $\dfrac{0 + 16}{6}$ or $\dfrac{G(2) - G(-4)}{6} = \dfrac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

Total for part (d) 2 points

Total for question 4 9 points

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5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

   (a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

   (b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

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(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

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Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

Model Solution	Scoring
(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.	
$f'(1) = \left.\frac{dy}{dx}\right\|_{(x, y)=(1, 4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2$.	Polynomial **1 point**
$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation **1 point**
(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.	
$f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x, y)=(1, 4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x, y)=(1.5, 4)}$	Euler's method with two steps **1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer **1 point**

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(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x \ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x \ln x$	**1 point**
$\ln\lvert y\rvert = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

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Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

	Model Solution	Scoring	
(a)	Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.		
	$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2$.	Polynomial	**1 point**
	$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation	**1 point**

Scoring notes:

- The first point is earned for $4 + \frac{4 \cdot \ln 1}{1!}(x-1)^1 + \frac{4}{2!}(x-1)^2$ or any correctly simplified equivalent expression. A term involving $(x-1)$ is not necessary. The polynomial must be written about (centered at) $x = 1$.
- If the first point is earned, the second point is earned for just "6" with no additional supporting work.
- If the polynomial is never explicitly written, the first point is not earned. In this case, to earn the second point supporting work of at least "$4 + 2(1)$" is required.

Total for part (a) 2 points

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(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1.5,\,4)}$	Euler's method with two steps	**1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for two steps (of size 0.5) of Euler's method, with at most one error. If there is any error, the second point is not earned.
- To earn the first point, a response must contain two Euler steps, $\Delta x = 0.5$, use of the correct expression for $\frac{dy}{dx}$, and use of the initial condition $f(1) = 4$.
  - The two Euler steps may be explicit expressions or may be presented in a table. Here is a minimal example of a (correctly labeled) table.

x	y	$\Delta y = \frac{dy}{dx} \cdot \Delta x$ or $\Delta y = \frac{dy}{dx} \cdot (0.5)$
1	4	0
1.5	4	$3 \ln 1.5$
2	$4 + 3 \ln 1.5$	

  - Note: In the presence of the correct answer, such a table does not need to be labeled in order to earn both points. In the presence of an incorrect answer, the table must be correctly labeled in order for the response to earn the first point.
- A single error in computing the approximation of $f(1.5)$ is not considered a second error if that incorrect value is imported correctly into an approximation of $f(2)$.
- Both points are earned for "$4 + 0.5 \cdot 0 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$" or "$4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$".
- Both points are earned for presenting the ordered pair $(2,\ 4 + 3 \ln 1.5)$ with sufficient supporting work.

Total for part (b) 2 points

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(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x \ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x \ln x$	**1 point**
$\ln\lvert y\rvert = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points. If an error in separation results in one side being correct ($\frac{1}{y}\,dy$ or $x \ln x\,dx$), the response is only eligible to earn the corresponding antiderivative point.
- The third point (antiderivative of $\frac{1}{y}$) can be earned for either $\ln y$ or $\ln\lvert y\rvert$.
- A response with no constant of integration can earn at most 3 out of 5 points.
- A response is eligible for the fourth point if it has earned the first point and at least 1 of the 2 antiderivative points.
- A response earns the fourth point by correctly including the constant of integration in an equation and then replacing x with 1 and y with 4.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (c) 5 points

Total for question 5 9 points

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6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

   (a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

   (b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

   (c) Determine the radius of convergence of the Maclaurin series for g.

   (d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

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(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

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Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

	Model Solution	Scoring	
(a)	State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.		
	e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions	**1 point**
	To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x}\,dx$ is finite (converges).	Improper integral	**1 point**
	$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^{b} e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big\vert_0^{b}\right) = \lim_{b\to\infty} \left(-e^{-b} + e^{0}\right) = 1$ Because the integral $\int_0^{\infty} e^{-x}\,dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation	**1 point**
(b)	Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.		
	$\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left\vert \frac{(-1)^n}{2e^n + 3} \right\vert} = \lim_{n\to\infty} \frac{2e^n + 3}{e^n} = 2$ The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \left\vert \frac{(-1)^n}{2e^n + 3} \right\vert$ converges by the limit comparison test. Thus, the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.	Sets up limit comparison	**1 point**
		Explanation	**1 point**

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(c) Determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{(-1)^n x^n} \right\rvert = \left\lvert \frac{(2e^n+3)x^{n+1}}{(2e^{n+1}+3)x^n} \right\rvert = \frac{2e^n+3}{2e^{n+1}+3}\lvert x \rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \frac{2e^n+3}{2e^{n+1}+3}\lvert x \rvert = \frac{1}{e}\lvert x \rvert$	Computes limit of ratio	**1 point**
$\frac{1}{e}\lvert x \rvert < 1 \Rightarrow \lvert x \rvert < e$ The radius of convergence is $R = e$.	Answer	**1 point**

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0. The alternating series error bound for the error of the approximation is the absolute value of the third term of the series. Error $\le \left\lvert \frac{(-1)^2}{2e^2+3} \right\rvert = \frac{1}{2e^2+3}$	Answer	**1 point**

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Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a)

State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

Model Solution	Scoring	
e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions	**1 point**
To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x}\,dx$ is finite (converges).	Improper integral	**1 point**
$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^{b} e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big\vert_0^{b}\right) = \lim_{b\to\infty} \left(-e^{-b} + e^{0}\right) = 1$ Because the integral $\int_0^{\infty} e^{-x}\,dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation	**1 point**

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Scoring notes:

- To earn the first point, a response must list all three conditions: e^{-x} is positive, decreasing, and continuous.
- The second point is earned for correctly writing the improper integral, or for presenting a correct limit equivalent to the improper integral, for example, $\lim_{b\to\infty}\int_0^b e^{-x}\,dx$.
- To earn the third point, a response must correctly use limit notation to evaluate the improper integral, find an evaluation of e^0 (or 1) and conclude that the integral converges or that the series converges.
- If an incorrect lower limit of 1 is used in the improper integral, then the second point is not earned. In this case, if the correct limit $(1/e)$ is presented, then the response is eligible for the third point.
- If the response only relies on using a geometric series approach, then no points are earned [0-0-0].
- A response that presents an evaluation with ∞, such as $e^{-\infty} = 0$, does not earn the third point.

Total for part (a) 3 points

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(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty}\frac{1}{e^n}$ to show that the series $g(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2e^n+3}$ converges absolutely.

$$\lim_{n\to\infty}\frac{\frac{1}{e^n}}{\left|\frac{(-1)^n}{2e^n+3}\right|}=\lim_{n\to\infty}\frac{2e^n+3}{e^n}=2$$

The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty}\frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{2e^n+3}\right|$ converges by the limit comparison test.

Thus, the series $g(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2e^n+3}$ converges absolutely.

Sets up limit comparison	**1 point**
Explanation	**1 point**

Scoring notes:

- The first point is earned for setting up the limit comparison, with or without absolute values. Limit notation is required to earn this point.
- The reciprocal of the given ratio is an acceptable alternative; the limit in this case is $1/2$.
- The second point cannot be earned without the use of absolute value symbols, which can occur explicitly or implicitly (e.g., a response might set up the limit comparison initially as $\lim_{n\to\infty}\frac{\frac{1}{e^n}}{\frac{1}{2e^n+3}}$).
- Earning the second point requires correctly evaluating the limit and noting that the limit is a positive number. For example, $L=2>0$ or $L=1/2>0$. Therefore, comparing the limit L to 1 does not earn the explanation point.
- A response does not have to repeat that $\sum_{n=0}^{\infty}\frac{1}{e^n}$ converges.
- A response that draws a conclusion based only on the sequence (such as $\frac{1}{e^n}$) without referencing a series does not earn the second point.
- If the response does not explicitly use the limit comparison test, then no points are earned in this part.
- A response cannot earn the second point for just concluding that "the series" converges absolutely because there are multiple series in this part of the problem. The response must specify that the series $g(1)$ or $\sum_{n=0}^{\infty}\frac{(-1)^n}{2e^n+3}$ converges absolutely.

Total for part (b) 2 points

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(c) Determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{(-1)^n x^n} \right\rvert = \left\lvert \frac{\left(2e^n+3\right)x^{n+1}}{\left(2e^{n+1}+3\right)x^n} \right\rvert = \frac{2e^n+3}{2e^{n+1}+3}\lvert x \rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \frac{2e^n+3}{2e^{n+1}+3}\lvert x \rvert = \frac{1}{e}\lvert x \rvert$	Computes limit of ratio	**1 point**
$\frac{1}{e}\lvert x \rvert < 1 \Rightarrow \lvert x \rvert < e$ The radius of convergence is $R = e$.	Answer	**1 point**

Scoring notes:

- The first point is earned for $\frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{(-1)^n x^n}$ or the equivalent. Once earned, this point cannot be lost.
- The second point cannot be earned without the first point.
- To be eligible for the third point, the response must have found a limit for a presented ratio such that the limiting value of the coefficient on $\lvert x \rvert$ is finite and not 0. The third point is earned for setting up an inequality such that the limit is less than 1, solving for $\lvert x \rvert$, and interpreting the result to find the radius of convergence.
- The radius of convergence must be explicitly presented, for example, $R = e$. The third point cannot be earned by presenting an interval, for example $-e < x < e$, with no identification of the radius of convergence.

Total for part (c) 3 points

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(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ decrease in magnitude to 0. The alternating series error bound for the error of the approximation is the absolute value of the third term of the series. $\text{Error} \le \left\lvert \frac{(-1)^2}{2e^2 + 3} \right\rvert = \frac{1}{2e^2 + 3}$	Answer **1 point**

Scoring notes:

- A response of $\frac{1}{2e^2 + 3}$ earns this point.

Total for part (d) 1 point

Total for question 6 9 points

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1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Model Solution	Scoring
(a) The total number of vehicles that arrive at the toll plaza from 6 A.M. to 10 A.M. is given by $\int_1^5 A(t)\,dt$.	1: Answer
(b) Average $= \frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966$	2: {1: Uses average value formula; 1: Answer}
(c) $A'(1) = 148.947272$ Because $A'(1) > 0$, the rate at which the vehicles arrive at the toll plaza is increasing.	2: {1: Considers $A'(1)$; 1: Answer with reason}
(d) $N'(t) = A(t) - 400 = 0$ $\Rightarrow A(t) = 400 \Rightarrow t = 1.469372,\ t = 3.597713$ $a = 1.469372$ $b = 3.597713$ t | $N(t) = \int_a^t (A(x) - 400)\,dx$ a | 0 b | 71.254129 4 | 62.338346 The greatest number of vehicles in line is 71.	4: {1: Considers $N'(1) = 0$; 1: $t = a$ and $t = b$; 1: Answer; 1: Justification}

Part A (AB or BC): Graphing calculator required
Question 1 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

	Model Solution	Scoring
(a)	Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).	
	The total number of vehicles that arrive at the toll plaza from 6 A.M. to 10 A.M. is given by $\int_1^5 A(t)\,dt$.	Answer **1 point**

Scoring notes:

- The response must be a definite integral with correct lower and upper limits to earn this point.
- Because $|A(t)| = A(t)$ for $1 \le t \le 5$, a response of $\int_1^5 \left|450\sqrt{\sin(0.62t)}\right|\,dt$ or $\int_1^5 |A(t)|\,dt$ earns the point.
- A response missing dt or using dx is eligible to earn the point.
- A response with a copy error in the expression for $A(t)$ will earn the point only in the presence of $\int_1^5 A(t)\,dt$.

Total for part (a) 1 point

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(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

$\text{Average} = \frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966$	Uses average value formula: $\frac{1}{b-a}\int_a^b A(t)\,dt$ **1 point**
The average rate at which vehicles arrive at the toll plaza from 6 A.M. to 10 A.M. is 375.537 (or 375.536) vehicles per hour.	Answer **1 point**

Scoring notes:

- The use of the average value formula, indicating that $a = 1$ and $b = 5$, can be presented in single or multiple steps to earn the first point. For example, the following response earns both points: $\int_1^5 A(t)\,dt = 1502.147865$, so the average value is 375.536966.
- A response that presents a correct integral along with the correct average value, but provides incorrect or incomplete communication, earns 1 out of 2 points. For example, the following response earns 1 out of 2 points: $\int_1^5 A(t)\,dt = 1502.147865 = 375.536966$.
- The answer must be correct to three decimal places. For example, $\frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966 \approx 376$ earns only the first point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{4}\int_1^5 A(t)\,dt = 79.416068$.
- Special case: $\frac{1}{5}\int_1^5 A(t)\,dt = 300.429573$ earns 1 out of 2 points.

Total for part (b) 2 points

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(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

$A'(1) = 148.947272$	Considers $A'(1)$	**1 point**
Because $A'(1) > 0$, the rate at which the vehicles arrive at the toll plaza is increasing.	Answer with reason	**1 point**

Scoring notes:

- The response need not present the value of $A'(1)$. The second line of the model solution earns both points.
- An incorrect value assigned to $A'(1)$ earns the first point (but will not earn the second point).
- Without a reference to $t = 1$, the first point is earned by any of the following:
  - 148.947 accurate to the number of decimals presented, with zero up to three decimal places (i.e., 149, 148, 148.9, 148.95, or 148.94)
  - $A'(t) = 148.947$ by itself
- To be eligible for the second point, the first point must be earned.
- To earn the second point, there must be a reference to $t = 1$.
- Degree mode: $A'(1) = 23.404311$

Total for part (c) 2 points

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

$N'(t) = A(t) - 400 = 0$ $\Rightarrow A(t) = 400 \Rightarrow t = 1.469372,\ t = 3.597713$ $a = 1.469372$ $b = 3.597713$	Considers $N'(t) = 0$	**1 point**
	$t = a$ and $t = b$	**1 point**
t | $N(t) = \int_a^t (A(x) - 400)\,dx$ a | 0 b | 71.254129 4 | 62.338346 The greatest number of vehicles in line is 71.	Answer	**1 point**
	Justification	**1 point**

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Scoring notes:

- It is not necessary to indicate that $A(t) = 400$ to earn the first point, although this statement alone would earn the first point.
- A response of "$A(t) \geq 400$ when $1.469372 \leq t \leq 3.597713$" will earn the first 2 points. A response of "$A(t) \geq 400$" along with the presence of exactly one of the two numbers above will earn the first point, but not the second. A response of "$A(t) \geq 400$" by itself will not earn either of the first 2 points.
- To earn the second point the values for a and b must be accurate to the number of decimals presented, with at least one and up to three decimal places. These may appear only in a candidates table, as limits of integration, or on a number line.
- A response with incorrect notation involving t or x is eligible to earn all 4 points.
- A response that does not earn the first point is still eligible for the remaining 3 points.
- To earn the third point, a response must present the greatest number of vehicles. This point is earned for answers of either 71 or $71.254***$ only.
- A correct justification earns the fourth point, even if the third point is not earned because of a decimal presentation error.
- When using a Candidates Test, the response must include the values for $N(a)$, $N(b)$, and $N(4)$ to earn the fourth point. These values must be correct to the number of decimals presented, with up to three decimal places. (Correctly rounded integer values are acceptable.)
- Alternate solution for the third and fourth points:

  For $a \leq t \leq b$, $A(t) \geq 400$. For $b \leq t \leq 4$, $A(t) \leq 400$.

  Thus, $N(t) = \int_a^t (A(x) - 400)\,dx$ is greatest at $t = b$.

  $N(b) = 71.254129$, and the greatest number of vehicles in line is 71.
- Degree mode: The response is only eligible to earn the first point because in degree mode $A(t) < 400$.

Total for part (d) 4 points

Total for question 1 9 points

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2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

   (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

   (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

   (c) Find the y-coordinate of the particle's position at time $t = 6$.

   (d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Model Solution	Scoring
(a) $\left.\frac{dy}{dx}\right\|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$	1: Answer
(b) $\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ $a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$	3: {1: Speed; 1: 1st component of acceleration; 1: 2nd component of acceleration}
(c) $y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right) dt = 5 + 6.570517 = 11.570517$	3: {1: Integrand; 1: Uses $y(4)$; 1: Answer}
(d) $\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 12.136228$	2: {1: Integrand; 1: Answer}

Part A (BC): Graphing calculator required
Question 2 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

	Model Solution	Scoring
(a)	Find the slope of the line tangent to the path of the particle at time $t = 4$.	
	$\left.\frac{dy}{dx}\right\|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$ The slope of the line tangent to the path of the particle at time $t = 4$ is 0.701.	Answer **1 point**

Scoring notes:

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: $\frac{y'(4)}{x'(4)} = 0.701$, $\frac{\ln\left(2 + 4^2\right)}{\sqrt{1 + 4^2}}$, or $\frac{\ln 18}{\sqrt{17}}$.
- Note: A response with an incorrect equation of the form "function $=$ constant", such as $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

Total for part (a) 1 point

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(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ The speed of the particle at time $t = 4$ is 5.035.	Speed	**1 point**
$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$ The acceleration vector of the particle at time $t = 4$ is $\langle 0.970, 0.444 \rangle$.	First component of acceleration	**1 point**
	Second component of acceleration	**1 point**

Scoring notes:

- To earn any of these points, the setup used to perform the calculation must be evident in the response. The following examples earn the first point: $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$ or $\sqrt{17 + (\ln 18)^2}$ and $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ would earn both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the x- and y-components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both $x''(t) = \frac{t}{\sqrt{1 + t^2}}$ and $y''(t) = \frac{2t}{2 + t^2}$, but which fails to evaluate both of these expressions at $t = 4$, earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

Total for part (b) 3 points

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(c) Find the y-coordinate of the particle's position at time $t = 6$.

$y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right) dt$	Integrand	**1 point**
	Uses $y(4)$	**1 point**
$= 5 + 6.570517 = 11.570517$ The y-coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570).	Answer	**1 point**

Scoring notes:

- For the first point, an integrand of $\ln\left(2 + t^2\right)$ can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for $y(4)$ must be added to a definite integral.
- A response that reports the correct x-coordinate of the particle's position at time $t = 6$ as $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2}\ dt = 11.200$ (or 11.201) instead of the y-coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
  - $y(6) = \int_4^6 \ln\left(2 + t^2\right)$ earns the first point and is eligible for the third.
  - $y(6) = \int_4^6 \ln\left(2 + t^2\right) + y(4)$ does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - $y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right)$ earns the first two points and is eligible for the third.

Total for part (c) 3 points

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(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

$\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	Integrand	**1 point**
$= 12.136228$ The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136.	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

Total for part (d) 2 points

Total for question 2 9 points

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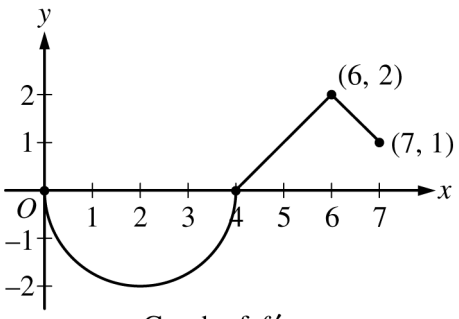

Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

   (a) Find $f(0)$ and $f(5)$.

   (b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

   (c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

   (d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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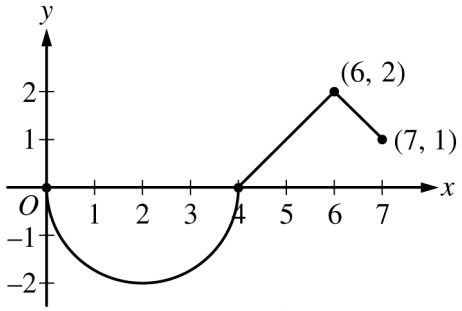


Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

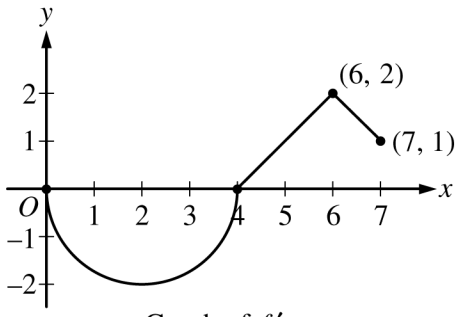
(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Model Solution	Scoring
(a) $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$	3: {1: Area of either region; 1: $f(0)$; 1: $f(5)$}
(b) The graph of f has a point of inflection at each of $x = 2$ and $x = 6$, because $f'(x)$ changes from decreasing to increasing at $x = 2$ and from increasing to decreasing at $x = 6$.	2: {1: Answer; 1: Justification}
(c) $g'(x) = f'(x) - 1$ $f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$ The graph of g is decreasing on the interval $0 \le x \le 5$ because $g'(x) \le 0$ on this interval.	2: {1: $g'(x) = f'(x) - 1$; 1: Interval with reason}
(d) g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$. Therefore, the absolute minimum occurs at $x = 5$, and $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the absolute minimum value of g.	2: {1: Considers $g'(x) = 0$; 1: Answer with justification}

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

Model Solution	Scoring	
$f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$	Area of either region – OR – $\int_0^4 f'(x)\,dx$ – OR – $\int_4^5 f'(x)\,dx$	**1 point**
	$f(0)$	**1 point**
	$f(5)$	**1 point**

Scoring notes:

- A response with answers of only $f(0) = \pm 2\pi$, or only $f(5) = \frac{1}{2}$, or both earns 1 of the 3 points.
- A response displaying $f(5) = \frac{7}{2}$ and a missing or incorrect value for $f(0)$ earns 2 of the 3 points.
- The second and third points can be earned in either order.
- Read unlabeled values from left to right and from top to bottom as $f(0)$ and $f(5)$. A single value must be labeled as either $f(0)$ or $f(5)$ in order to earn any points.

Total for part (a) 3 points

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(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

The graph of f has a point of inflection at each of $x = 2$ and $x = 6$, because $f'(x)$ changes from decreasing to increasing at $x = 2$ and from increasing to decreasing at $x = 6$.	Answer	**1 point**
	Justification	**1 point**

Scoring notes:

- A response that gives only one of $x = 2$ or $x = 6$, along with a correct justification, earns 1 of the 2 points.
- A response that claims that there is a point of inflection at any value other than $x = 2$ or $x = 6$ earns neither point.
- To earn the second point a response must use correct reasoning based on the graph of f'. Examples of correct reasoning include:
  - Correctly discussing the signs of the slopes of the graph of f'
  - Citing $x = 2$ and $x = 6$ as the locations of local extrema on the graph of f'
- Examples of reasoning not (sufficiently) connected to the graph of f' include:
  - Reasoning based on sign changes in f'' unless the connection is made between the sign of f'' and the slopes of the graph of f'
  - Reasoning based only on the concavity of the graph of f
- The second point cannot be earned by use of vague or undefined terms such as "it" or "the function" or "the derivative."
- Responses that report inflection points as ordered pairs must report the points $(2, 3 + \pi)$ and $(6, 5)$ in order to earn the first point. If the y-coordinates are reported incorrectly, the response remains eligible for the second point.

Total for part (b) 2 points

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

$g'(x) = f'(x) - 1$	$g'(x) = f'(x) - 1$	**1 point**
$f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$ The graph of g is decreasing on the interval $0 \le x \le 5$ because $g'(x) \le 0$ on this interval.	Interval with reason	**1 point**

Scoring notes:

- The first point can be earned for $f'(x) \le 1$ or the equivalent, in words or symbols.
- Endpoints do not need to be included in the interval to be eligible for the second point.

Total for part (c) 2 points

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(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$. Therefore, the absolute minimum occurs at $x = 5$, and $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the absolute minimum value of g.	Considers $g'(x) = 0$	**1 point**
	Answer with justification	**1 point**

Scoring notes:

- A justification that uses a local argument, such as "g' changes from negative to positive (or g changes from decreasing to increasing) at $x = 5$" must also state that $x = 5$ is the only critical point.
- If $g'(x) = 0$ (or equivalent) is not declared explicitly, a response that isolates $x = 5$ as the only critical number belonging to $(0, 7)$ earns the first point.
- A response that imports $g'(x) = f'(x)$ from part (c) is eligible for the first point but not the second.
  - In this case, consideration of $x = 4$ as the only critical number on $(0, 7)$ earns the first point.
- Solution using Candidates Test:

  $g'(x) = f'(x) - 1 = 0 \Rightarrow x = 5,\ x = 7$

x	$g(x)$
0	$3 + 2\pi$
5	$-\frac{3}{2}$
7	$-\frac{1}{2}$

  The absolute minimum value of g on the interval $0 \le x \le 7$ is $-\frac{3}{2}$.

- When using a Candidates Test, a response may import an incorrect value of $f(0) = g(0) > -\frac{3}{2}$ from part (a). The second point can only be earned for an answer of $-\frac{3}{2}$.

Total for part (d) 2 points

Total for question 3 9 points

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t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

   (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

   (b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

   (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

   (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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NO CALCULATOR ALLOWED

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

   (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

   (b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

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NO CALCULATOR ALLOWED

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

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t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

   (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

   (b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

   (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

   (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Model Solution	Scoring
(a) $r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{10 - 7}$ $= \frac{0.6}{3} = 0.2$ centimeter per day per day	2: {1: Answer with work; 1: Units}
(b) $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ $r(t)$ is twice-differentiable. $\Rightarrow r'(t)$ is differentiable. $\Rightarrow r'(t)$ is continuous. Therefore, by the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$.	2: {1: $r'(0) < -6 < r'(3)$; 1: Conclusion using IVT}
(c) $\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$ $= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5) = -51$	2: {1: Form of right Riemann sum; 1: Answer}
(d) $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$ $\left.\frac{dV}{dt}\right\|_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$	3: {1: Product rule; 1: Chain rule; 1: Answer}

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

	Model Solution	Scoring
(a)	Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.	
	$r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{10 - 7}$	$r''(8.5)$ with supporting work **1 point**
	$= \dfrac{0.6}{3} = 0.2$ centimeter per day per day	Units **1 point**

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as cm/day^2).

Total for part (a) 2 points

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(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow$ $r'(t)$ is differentiable. $\Rightarrow$ $r'(t)$ is continuous. $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$.	$r'(0) < -6 < r'(3)$	**1 point**
	Conclusion using Intermediate Value Theorem	**1 point**

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not state that -6 is between -6.1 and -5. Thus this response does not earn the first point.
- To earn the second point:
  - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
  - The response must have earned the first point.
    - Exception: A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
  - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of "yes" would be sufficient.)
- To earn the second point, the response need not explicitly name the IVT, but if a theorem is named, it must be correct.

Total for part (b) 2 points

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(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

$\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	**1 point**
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	**1 point**

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15,\ -17.6,\ -11.4,\ -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t)\,dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t)\,dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

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(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$	Product rule	**1 point**
	Chain rule	**1 point**
$\left.\frac{dV}{dt}\right\|_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$ The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.	Answer	**1 point**

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$ earn any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point, but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points

Total for question 4 9 points

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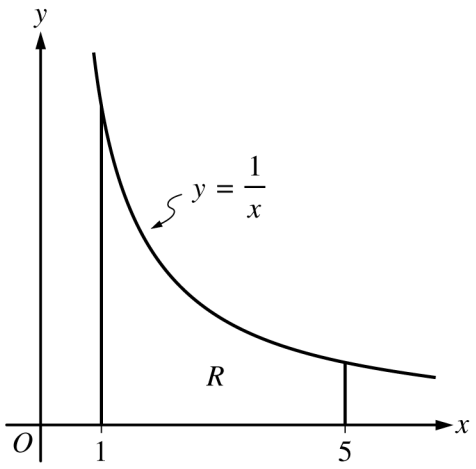

Figure 1

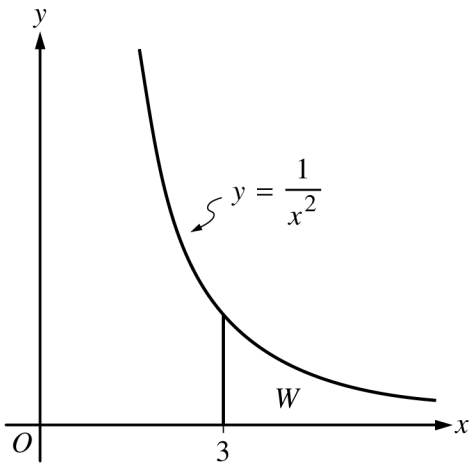

Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

   (a) Find the area of region R.

   (b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

   (c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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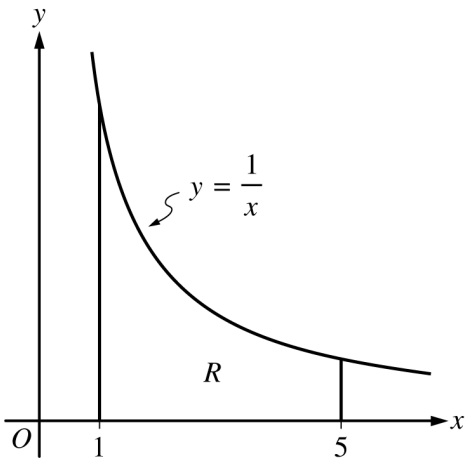

Figure 1

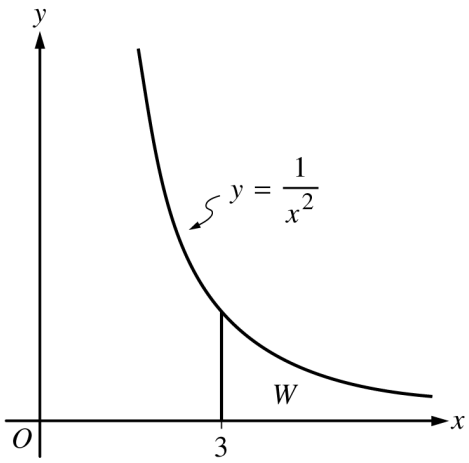

Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

   (a) Find the area of region R.

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NO CALCULATOR ALLOWED

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

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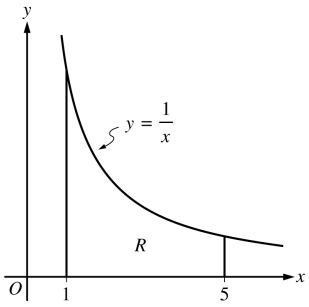

Figure 1

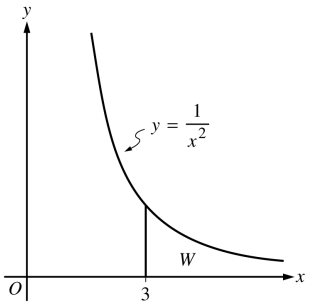

Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Model Solution	Scoring
(a) Area $= \int_1^5 \frac{1}{x}\,dx = \ln x\big\|_1^5 = \ln 5 - \ln 1 = \ln 5$	2: {1: Integral; 1: Answer}
(b) Volume $= \int_1^5 xe^{x/5}\,dx$ $\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C = 5e^{x/5}(x-5) + C$ Volume $= 5e^{x/5}(x-5)\big\|_1^5 = 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$ $u = x \quad dv = e^{x/5}\,dx$ $du = dx \quad v = 5e^{x/5}$	4: {1: Definite integral; 1: u and dv; 1: $\int xe^{x/5}dx = 5xe^{x/5} - \int 5e^{x/5}dx$; 1: Answer}
(c) Volume $= \pi\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty}\int_3^b \frac{1}{x^4}\,dx = \pi \lim_{b\to\infty}\left(\frac{1}{-3x^3}\Big\|_3^b\right)$ $= \pi \lim_{b\to\infty}\left(\frac{1}{-3}\right)\left[\frac{1}{b^3} - \frac{1}{3^3}\right] = \pi\left(\frac{1}{-3}\right)\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$	3: {1: Improper integral; 1: Antiderivative; 1: Answer}

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

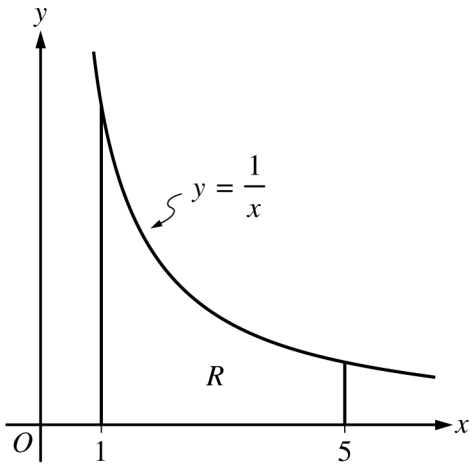

Figure 1

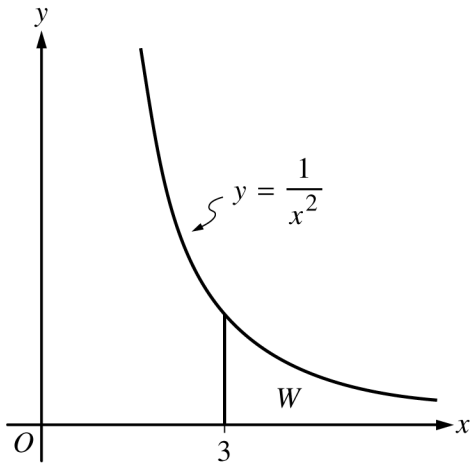

Figure 2

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

	Model Solution	Scoring	
(a)	Find the area of region R.		
	$\text{Area} = \int_1^5 \frac{1}{x}\,dx$	Integral	**1 point**
	$= \ln x\big\|_1^5$ $= \ln 5 - \ln 1 = \ln 5$	Answer	**1 point**

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Scoring notes:

- A definite integral with incorrect bounds does not earn either point.
- An unevaluated indefinite integral does not earn either point.
- An indefinite integral which is evaluated in a later step may earn one or both points. For example, $\int \frac{1}{x}\,dx = \ln 5 - \ln 1$ (or $\ln 5$) does not earn the first point but does earn the second. However, $\int \frac{1}{x}\,dx = \ln x + C \Rightarrow$ Area $= \ln 5 - \ln 1$ earns both points.

Total for part (a) 2 points

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

Volume $= \int_1^5 xe^{x/5}\,dx$	Definite integral	**1 point**
Using integration by parts, $u = x \quad dv = e^{x/5}\,dx$ $du = dx \quad v = 5e^{x/5}$	u and dv	**1 point**
$\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C$ $= 5e^{x/5}(x-5) + C$	$\int xe^{x/5}\,dx$ $= 5xe^{x/5} - \int 5e^{x/5}\,dx$	**1 point**
Volume $= 5e^{x/5}(x-5)\Big\|_1^5$ $= 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$	Answer	**1 point**

Scoring notes:

- The first point is earned for $c\int_1^5 xe^{x/5}\,dx$, where $c \neq 0$. Errors of $c \neq 1$, for example $c = \pi$, will not earn the fourth point.
- Incorrect integrals that require integration by parts are still eligible for the second and third points. Both of these points will be earned with at least one correct application of integration by parts.
- The second point will be earned with an implied u and dv in the presence of $5xe^{x/5} - \int 5e^{x/5}\,dx$.
- The tabular method may be used to show integration by parts. In this case, the second point is earned by having columns (labeled or unlabeled) that begin with x and $e^{x/5}$. The third point is earned for either $5xe^{x/5} - \int 5e^{x/5}\,dx$ or $5xe^{x/5} - 25e^{x/5}$.
- Limits of integration may be present, omitted, or partially present in the work for the second and third points.
- The fourth point is earned only for the correct answer.

Total for part (b) 4 points

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(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

$\text{Volume} = \pi\displaystyle\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty}\int_3^b \frac{1}{x^4}\,dx$	Improper integral	**1 point**
$= \pi \lim_{b\to\infty}\left(\left.\frac{1}{-3x^3}\right\|_3^b\right)$	Antiderivative	**1 point**
$= \pi \lim_{b\to\infty}\left(\frac{1}{-3}\right)\left[\frac{1}{b^3} - \frac{1}{3^3}\right]$ $= \pi\left(\frac{1}{-3}\right)\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $c\displaystyle\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx$ or $\lim_{b\to\infty} c\int_3^b \frac{1}{x^4}\,dx$, where $c \neq 0$. Errors of $c \neq \pi$ will not earn the third point.
- The second point is earned for a correct antiderivative of any integrand of the form $\frac{1}{x^n}$, for any integer $n \geq 2$.
- To earn the answer point, a response must use correct limit notation and cannot include arithmetic with infinity such as $\frac{1}{\infty^3}$.

Total for part (c) 3 points

Total for question 5 9 points

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6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

   (a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

   (b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

   (c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

   (d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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NO CALCULATOR ALLOWED

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

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NO CALCULATOR ALLOWED

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

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6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Model Solution	Scoring
(a) $\lim_{n\to\infty} \left\lvert \frac{\frac{(-1)^{n+1} x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right\rvert = \lim_{n\to\infty} \left\lvert \frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}} \right\rvert$ $= \lim_{n\to\infty} \left\lvert x^2 \left(\frac{2n+1}{2n+3}\right) \right\rvert = \lvert x^2 \rvert \qquad \lvert x^2 \rvert < 1$ for $\lvert x \rvert < 1$. The series converges when $-1 < x < 1$. $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$ $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$ Convergent by the Alternating Series Test The interval of convergence is $-1 \le x \le 1$	4: {1: Sets up ratio 1: $-1 < x < 1$ 1: Considers both endpoints 1: Analysis and interval of convergence
(b) $\left\lvert f\left(\frac{1}{2}\right) - \frac{1}{2} \right\rvert < \left\lvert \frac{(-1)^1 \left(\frac{1}{2}\right)^3}{3} \right\rvert = \frac{1}{24} < \frac{1}{10}$ The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. By the alternating series error bound, the error is less then $\frac{1}{24} < \frac{1}{10}$.	2: {1: Uses the second term 1: Justification
(c) $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	2: {1: First four terms 1: General term
(d) $f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right)$ is a geometric series with $a = 1$ and $r = -\frac{1}{36}$. $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	1: Answer

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

	Model Solution	Scoring	
(a)	Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.		
	$\lim_{n\to\infty}\left\lvert\frac{\frac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert x^2\left(\frac{2n+1}{2n+3}\right)\right\rvert = \lvert x^2\rvert$	Sets up ratio	**1 point**
	$\lvert x^2\rvert < 1$ for $\lvert x\rvert < 1$. The series converges when $-1 < x < 1$.	Identifies interior of interval of convergence	**1 point**
	When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$.	Considers both endpoints	**1 point**
	The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1 \le x \le 1$.	Analysis and interval of convergence	**1 point**

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Scoring notes:

- A response that includes the substitution error of the form $x^{2(n+1)+1} = x^{2n+3}$ appearing as $x^{2n+1+1} = x^{2n+2}$ in setting up a ratio is eligible for the first 3 points but does not earn the fourth point.
- The first point is earned by presenting a correct ratio with or without absolute values.
- To earn the second point a response must:
  - use the absolute value of the ratio, or resolve the lack of absolute values by concluding $x^2 < 1$ (without any errors), and correctly evaluate the limit of the ratio, including correct limit notation, and
  - identify the interior of the interval of convergence. The response can use either interval notation or the compound inequality $-1 < x < 1$ ($|x| < 1$ is insufficient).
- The only incorrect interval eligible for the third point is $0 < x < 1$. In this case, to earn the third point, the response needs to evaluate the general term at $x = 1$.

Total for part (a) 4 points

(b) Show that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. Justify your answer.

The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. The first term of the series for $f\left(\frac{1}{2}\right)$ is $\frac{1}{2}$. Using the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most the absolute value of the second term of the series.

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \left|\frac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right| = \frac{1}{24} < \frac{1}{10}$$

Uses second term	**1 point**
Justification	**1 point**

Scoring notes:

- The first point is earned by correctly using $x = \frac{1}{2}$ in the second term (listing the second term as part of a polynomial is insufficient). Using $x = \frac{1}{2}$ in any term of degree five or higher does not earn this point.
- To earn the second point a response must:
  - have earned the first point,
  - state that the series is alternating and that its terms decrease to zero, and
  - present the inequality $\text{Error} < \frac{1}{24} < \frac{1}{10}$ (or the equivalent).
- A response that states $\text{Error} = \frac{1}{24}$ does not earn the second point.

Total for part (b) 2 points

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(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	First four terms	**1 point**
	General term	**1 point**

Scoring notes:

- The first point is earned by presenting the first four nonzero terms in a list or as part of a polynomial or series.
- The second point is earned by identifying the general term (either individually or as part of a polynomial or series).
- Read "$=$" as "$\approx$" as necessary.

Total for part (c) 2 points

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right)$ is a geometric series with $a = 1$ and $r = -\frac{1}{36}$. $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	Answer	**1 point**

Scoring notes:

- The result from part (c) must be geometric in order to be eligible for this point.
- If a response imports an incorrect geometric series from part (c), this point is earned only for a consistent answer.

Total for part (d) 1 point

Total for question 6 9 points

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t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

   Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

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t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

(a)

$\int_{60}^{135} f(t)dt$ is the number of gallons pumped from $t = 60$ to $t = 135$ seconds.

$\int_{60}^{135} f(t)dt \approx (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$ gal

3: {1: interpretation with units; 1: Form of Right Riemann sum; 1: Answer}

(b)

Yes, because f is differentiable and therefore continuous and the average rate of change on $[60, 120]$ is $\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$

The Mean Value Theorem guarantees there exists a value of c, $60 < c < 120$, such that $f'(c) = \frac{f(120) - f(60)}{120 - 60} = 0$.

2: {1: $f(120) - f(60) = 0$; 1: answer with justification}

(c)

average rate of flow $= \frac{1}{150}\int_0^{150} g(t)dt = 0.09599\ldots$ gal per sec

2: {1: average value formula; 1: answer}

(d) $g'(140) = -0.004908\ldots$ The rate of flow of gasoline is decreasing at a rate of 0.004908 ... gallons per second per second at $t = 140$ seconds.

2: {1: $g'(140)$; 1: interpretation}

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Part A (AB or BC): Graphing calculator required
Question 1 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

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Model Solution	Scoring

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$.

$\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.	Interpretation with units	**1 point**
$\int_{60}^{135} f(t)\,dt$ $\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120)$ $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$	Form of Riemann sum	**1 point**
	Answer	**1 point**

Scoring notes:

- To earn the first point, the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point, at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.
- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30), 0.1(30), 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90-60) + f(120)(120-90) + f(135)(135-120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only "8.25" does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

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(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow$ f is continuous on $[60, 120]$. $\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$ By the Mean Value Theorem, there must exist a c, for $60 < c < 120$, such that $f'(c) = 0$.	$f(120) - f(60) = 0$	**1 point**
	Answer with justification	**1 point**

Scoring notes:

- To earn the first point, a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must
  - have earned the first point,
  - state that f is continuous because f is differentiable (or equivalent),
  - answer "yes" in some way.
- A response may reference either the Mean Value Theorem or Rolle's Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

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(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.

$\frac{1}{150-0}\int_0^{150} g(t)\,dt$	Average value formula	**1 point**
$= 0.0959967$ The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).	Answer	**1 point**

Scoring notes:

- The exact value of $\frac{1}{150}\int_0^{150} g(t)\,dt$ is $\frac{12}{125}\sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t)\,dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication, (e.g. $\int_0^{150} g(t)\,dt = \frac{14.399504}{150} = 0.0959967$), earns 1 out of 2 points.
- A response of $\int_0^{150} g(t)\,dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150}\int_0^{150} g(t)\,dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

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(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$ $g'(140) = -0.005$ (or -0.004)	$g'(140)$	**1 point**
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation	**1 point**

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500}\cos\left(\frac{49}{36}\right) - \frac{49}{9000}\sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point, a response must present some numerical value for $g'(140)$.
- To earn the second point, the interpretation must include "the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]" and "at $t = 140$" (or equivalent).
- An interpretation of "decreasing at a rate of -0.005" or "increasing at a rate of 0.005" does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points

Total for question 1 9 points

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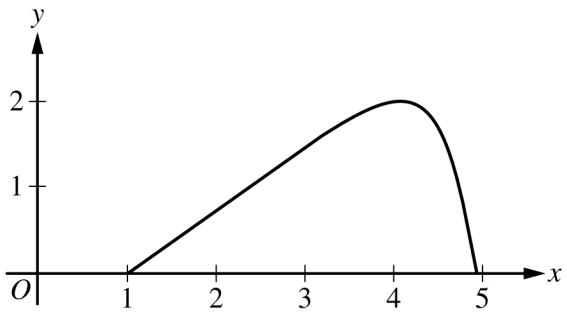

2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

   (a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

   (b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

   (c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

   (d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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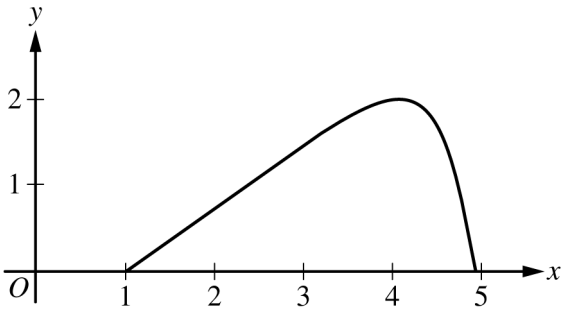

2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

(a) acceleration vector: $\langle x''(1), y''(1)\rangle = \langle -1.4444\ldots, -1.6829\ldots\rangle$

2: $\begin{cases} 1: x''(1) \text{ with set up} \\ 1: y''(1) \text{ with set up} \end{cases}$

(b) $\text{speed}(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = 1.5 \Rightarrow t = 1.2544\ldots$

2: $\begin{cases} 1: \sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5 \\ 1: \text{answer} \end{cases}$

(c) slope of tangent line: $\frac{y'(1)}{x'(1)} = 0.6295\ldots$

$x(1) = \underbrace{1}_{x(0)} + \int_0^1 x'(t)dt = 3.3415\ldots$

3: $\begin{cases} 1: \text{slope with. supporting work} \\ 1: \int_0^1 e^{\cos t}\, dt \\ 1: x(1) \end{cases}$

(d)

$\int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2}dt = 6.0346\ldots$

2: $\begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

	Model Solution	Scoring	
(a)	Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.		
	$x''(1) = \dfrac{d}{dt}\left(e^{\cos t}\right)\Big\|_{t=1} = -1.444407$ $y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$ $y''(1) = \dfrac{d}{dt}(2\cos t)\Big\|_{t=1} = -1.682942$ The acceleration vector at time $t = 1$ is $a(1) = \langle -1.444, -1.683 \text{ (or } -1.682)\rangle$.	$x''(1)$ with setup	**1 point**
		$y''(1)$ with setup	**1 point**

Scoring notes:

- The exact answer is $\langle x''(1), y''(1)\rangle = \left\langle -e^{\cos 1}\sin 1, -2\sin 1\right\rangle$.
- $\left\langle -e^{\cos t}\sin t, -2\sin t\right\rangle$ together with an incorrect or missing evaluation at $t = 1$ earns 1 of the 2 points.
- A response that equates an expression to a numerical value, such as, $\left\langle -e^{\cos t}\sin t, -2\sin t\right\rangle = \left\langle -e^{\cos 1}\sin 1, -2\sin 1\right\rangle$, earns 1 of the 2 points.
- An unsupported correct acceleration vector earns 1 of the 2 points.
- The acceleration vector may be presented with other symbols, for example $(\ ,\ )$ or $[\ ,\ ]$.
- The components may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode, or the question is made simpler by using degree mode).

  In degree mode, $x''(1) = -0.000828$ or -0.047433 and $y''(1) = -0.000609$ or -0.034905. A response that presents one of these values with correct setups earns 1 of the 2 points.

Total for part (a) 2 points

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(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}$ $0 \le t \le \pi$ and $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$	$\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$	**1 point**
$\Rightarrow t = 1.254472,\ t = 2.358077$ The first time at which the speed of the particle is 1.5 is $t = 1.254$.	Answer	**1 point**

Scoring notes:

- A response with an implied equation is eligible for both points. For example, a response of "Speed $= \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}$ and is first equal to 1.5 at $t = 1.254$" earns both points.
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$ earns the first point. Speed $= 1.5$ by itself does not earn the first point. Both of these responses are eligible to earn the second point.
- A response need not consider the value $t = 2.358077$.
- A response of $t = 1.254$ alone does not earn either point.
- A response with a parenthesis error(s) in either $\left(e^{\cos t}\right)^2$ or $(2\cos t)^2$ does not earn the first point but does earn the second point for the correct answer. Note: $\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$ is not considered a parenthesis error.
- Degree mode: In degree mode, $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ has no solution for $0 \le t \le \pi$.
  A response that finds no time t at which the speed of the particle is 1.5 cannot be assumed to be working in degree mode.

Total for part (b) 2 points

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(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$ $\left.\frac{dy}{dx}\right\|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530$ The slope of the line tangent to the curve at $t = 1$ is 0.630 (or 0.629).	Slope with supporting work	**1 point**
$x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575$ The x-coordinate of the position at $t = 1$ is 3.342 (or 3.341).	$\int_0^1 e^{\cos t}\,dt$ $x(1)$	**1 point** **1 point**

Scoring notes:

- To earn the first point, the response must communicate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; for example:
  - $\frac{dy}{dx} = \frac{2\cos 1}{e^{\cos 1}}$
  - $x'(1) = 1.716526,\ y'(1) = 1.080605$ slope $= 0.63$,
  - $\frac{dy/dt}{dx/dt} = 0.63$
  - $\frac{dy}{dt} = 2\cos t,$ slope $= 0.63$
- A response may import an incorrect expression for $y'(t)$ or value of $y'(1)$ from part (a), provided it was declared in part (a).
- The second point is earned for a response that presents the definite integral $\int_0^1 e^{\cos t}\,dt$ with or without the initial condition.
- For the second point, if the differential is missing:
  - $\int_0^1 e^{\cos t}$ earns the second point and is eligible for the third point.
  - $x(1) = \int_0^1 e^{\cos t}$ earns the second point but is not eligible for the third point.
  - $x(1) = 1 + \int_0^1 e^{\cos t}$ earns the second point and is eligible for the third point.
  - $x(1) = \int_0^1 e^{\cos t} + 1$ does not earn the second point but earns the third point for the correct answer.
- The third point is not earned for a response that presents an incorrect statement, such as.

  $x(1) = \int_0^1 e^{\cos t}\,dt = 1 + 2.342.$
- Degree mode: In degree mode, $\frac{dy}{dx} = 0.735759$ or 0.012841 and $1 + \int_0^1 e^{\cos t}\,dt = 3.718144$.

Total for part (c) 3 points

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(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

$\int_0^{\pi} \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}\ dt$	Integral	**1 point**
$= 6.034611$ The total distance traveled by the particle over $0 \le t \le \pi$ is 6.035 (or 6.034).	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- Parentheses errors were assessed in part (b) and, therefore, will not affect the scoring in part (d).
- If the integrand is an incorrect speed function imported from part (b), the response earns the first point and does not earn the second point.
- An unsupported answer of 6.035 (or 6.034) does not earn either point.
- Degree mode: In degree mode, the total distance is 10.596835 or 8.536161.

Total for part (d) 2 points

Total for question 2 9 points

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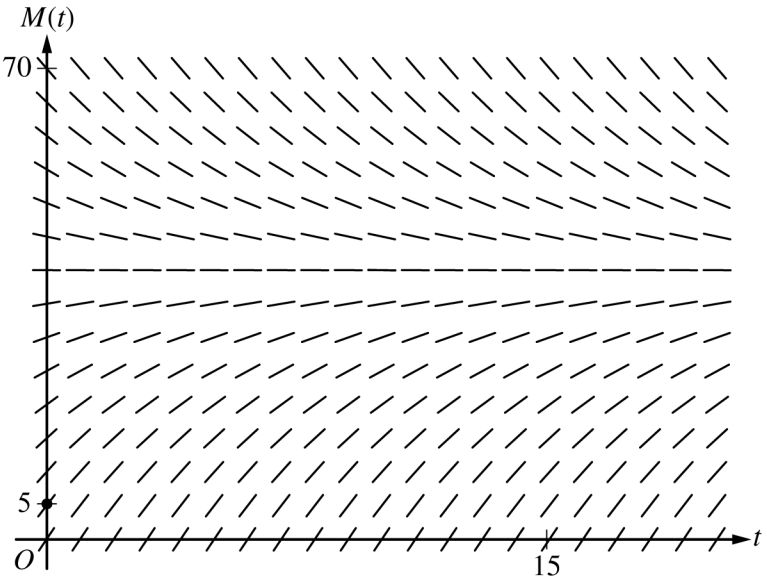
3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

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NO CALCULATOR ALLOWED

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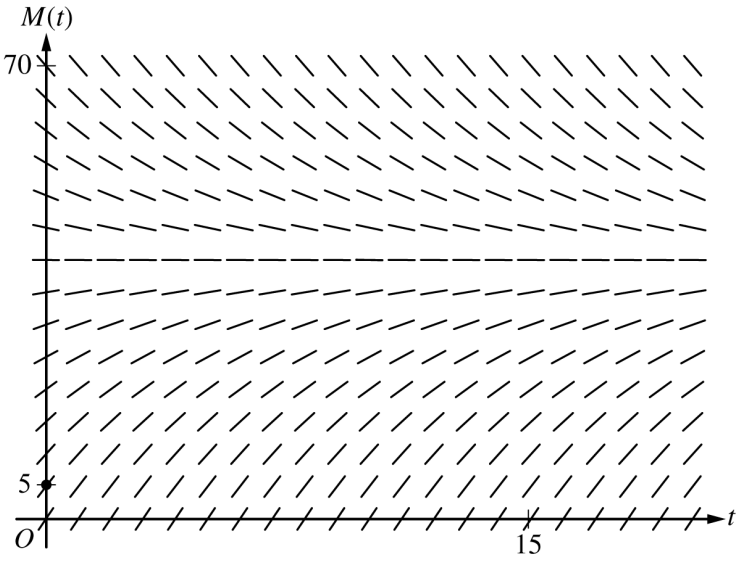

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

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NO CALCULATOR ALLOWED

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

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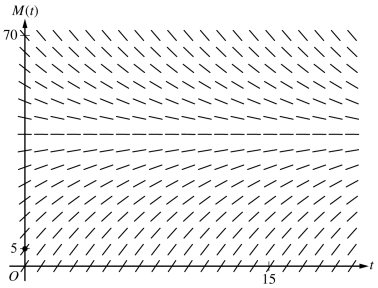

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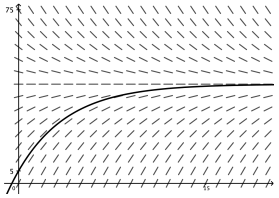
(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

(a) 

(b) $M(t) \approx 5 + \left.\frac{dM}{dt}\right|_{(0,5)} t =$
$5 + \frac{1}{4}(40 - 5)t = 5 + \frac{35}{4}t$
$M(2) \approx 5 + \frac{35}{4}(2) = 5 + \frac{35}{2} = \frac{45}{2}\ °C$

(c) $\frac{d^2M}{dt^2} = \frac{d}{dt}\left(\frac{1}{4}(40 - M)\right) = \frac{1}{4}\left(-\frac{dm}{dt}\right) = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$
$\left.\frac{d^2M}{dt^2}\right|_{(0,5)} = -\frac{1}{4}\left(\frac{1}{4}(40 - 5)\right) = -\frac{35}{16} < 0$ The approximation from part (b) is an overestimate because $M(t)$ is concave down near $t = 0$.

(d) $\int \frac{1}{40-M} dM = \int \frac{1}{4} dt \Rightarrow -\ln|40 - M| = \frac{1}{4}t + C$
$(0,5) \Rightarrow -\ln|40 - 5| = \frac{1}{4}(0) + C \Rightarrow C = -\ln 35$
$\ln|40 - M| = -\frac{1}{4}t + \ln 35 \Rightarrow |40 - M| = 35e^{-\frac{1}{4}t}$
$M = 40 \pm 35e^{-\frac{1}{4}t} \quad M(0) = 5 \Rightarrow M = 40 - 35e^{-\frac{1}{4}t}$

1: solution curve

2: 1: $\left.\frac{dM}{dt}\right|_{t=0}$
1: approximation

2: 1: $\frac{d^2M}{dt^2}$
1: overestimate with reason

4: 1: separates variables
1: Find antiderivatives
1: Constant of integration and uses initial condition
1: solves for $M(t)$

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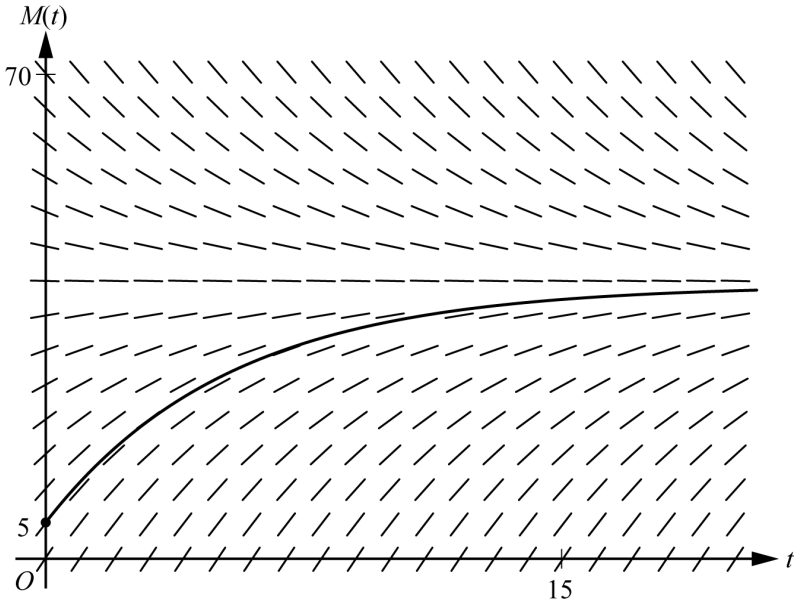
Part A (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5°\text{C}$. It can be shown that $M(t) < 40$ for all values of t.

	Model Solution	Scoring
(a)	A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.	
		Solution curve **1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total for part (a) 1 point

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

$\left.\frac{dM}{dt}\right\|_{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$	$\left.\frac{dM}{dt}\right\|_{t=0}$	**1 point**
The tangent line equation is $y = 5 + \frac{35}{4}(t - 0)$. $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.	Approximation	**1 point**

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
  - passes through the point $(0, 5)$, and
  - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b) 2 points

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(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

$\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$	$\frac{d^2M}{dt^2}$	**1 point**
Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason	**1 point**

Scoring notes:

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
    - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point, but is eligible to earn the second point for a consistent answer and reason
- To earn the second point a response must include one of $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

Total for part (c) 2 points

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(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\frac{dM}{40 - M} = \frac{1}{4}dt$ $\int \frac{dM}{40 - M} = \int \frac{1}{4}\, dt$	Separates variables	**1 point**
$-\ln\lvert 40 - M \rvert = \frac{1}{4}t + C$	Finds antiderivatives	**1 point**
$-\ln\lvert 40 - 5 \rvert = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow \lvert 40 - M \rvert = 40 - M$ $-\ln(40 - M) = \frac{1}{4}t - \ln 35$ $\ln(40 - M) = -\frac{1}{4}t + \ln 35$	Constant of integration and uses initial condition	**1 point**
$40 - M = 35e^{-t/4}$ $M = 40 - 35e^{-t/4}$	Solves for M	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is sufficient.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
    - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M.
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d) 4 points

Total for question 3 9 points

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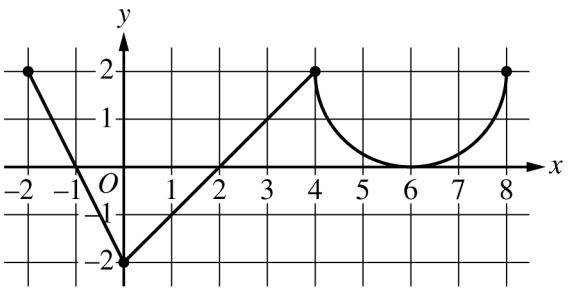

Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

   (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

   (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

   (c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

   (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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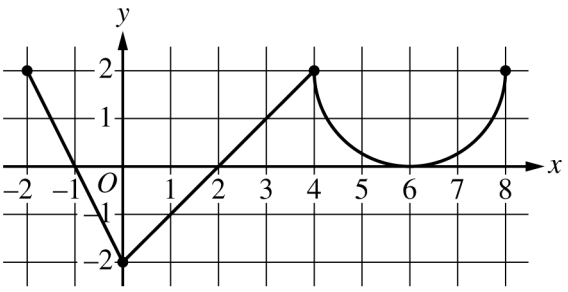

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   (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

   (c) Find the value of $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

   (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

(a) Neither because f' does not change signs at $x = 6$.

1: answer with reason

(b) f is concave down on $(-2,0)$ and $(4,6)$ because f' is decreasing.

2: $\begin{cases} 1: \text{intervals} \\ 1: \text{Reason} \end{cases}$

(c) $\lim_{x\to 2}[6f(x) - 3x] = 6f(2) - 6 = 6(1) - 6 = 0$

f is differentiable at $x = 2$, so f is continuous at $x = 2$,

so $\lim_{x\to 2} f(x) = f(2) = 1$

$\lim_{x\to 2}[x^2 - 5x + 6] = 2^2 - 10 + 6 = 0$

$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \underbrace{\lim_{x\to 2} \frac{6f'(x)-3}{2x-5}}_{\substack{\text{l'hospital's} \\ \text{rule}}} = \frac{6f'(2)-3}{4-5} = \frac{6(0)-3}{-1} = \frac{-3}{-1} = 3$

3: $\begin{cases} 1: \text{Limits of numerator and denomonator} \\ 1: \text{Uses L'Hospital's Rule} \\ 1: \text{Answer} \end{cases}$

(d) The absolute minimum value of f occurs at an endpoint of the interval or a critical point. $f'(x) = 0 \Rightarrow x = -1, 2$ and 6

x	$f(x)$
-2	$f(2) - f(-2) = \int_{-2}^{2} f'(x)dx \Rightarrow f(-2) = f(2) - \int_{-2}^{2} f'(x)dx$ $f(-2) = 1 - \left[\frac{1}{2}(2)(1) + \frac{1}{2}(3)(-2)\right] = 1 - [1 - 3] = 3$
-1	$f(-1) = f(2) - \int_{-1}^{2} f'(x)dx = 1 - \left[\frac{1}{2}(3)(-2)\right] = 4$
2	1 Absolute minimum by Candidate's Test
6	$f(6) - f(2) = \int_{2}^{6} f'(x)dx \Rightarrow f(6) = f(2) + \int_{2}^{6} f'(x)dx$ $f(6) = 1 + \left[\frac{1}{2}(2)(2) + \left[(2)(2) - \frac{1}{4}\pi(2^2)\right]\right] = 7 - \pi \approx 4$
8	$f(8) = f(2) + \int_{2}^{8} f'(x)dx$ $f(8) = 1 + \left[\frac{1}{2}(2)(2) + \left[(4)(2) - \frac{1}{2}\pi(2^2)\right]\right] = 11 - 2\pi \approx 5$

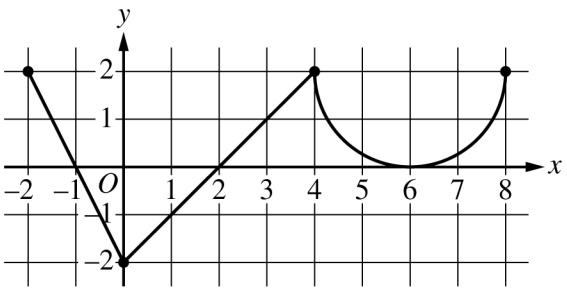
3: $\begin{cases} 1: \text{Considers } f'(x) = 0 \\ 1: \text{Justification} \\ 1: \text{answer} \end{cases}$

Part A (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

	Model Solution	Scoring
(a)	Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.	
	$f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$ $f'(x)$ does not change sign at $x = 6$, so there is neither a relative maximum nor a relative minimum at this location.	Answer with reason **1 point**

Scoring notes:

- A response that declares $f'(x)$ does not change sign at $x = 6$, so neither, is sufficient to earn the point.
- A response does not have to present intervals on which $f'(x)$ is positive or negative, but if any are given, they must be correct. Any presented intervals may include none, one, or both endpoints.
- A response that declares $f'(x) > 0$ before and after $x = 6$ does not earn the point.

Total for part (a) 1 point

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(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals $(-2, 0)$ $(4, 6)$.	Intervals	**1 point**
	Reason	**1 point**

Scoring notes:

- The first point is earned only by an answer of $(-2, 0)$ and $(4, 6)$, or these intervals including one or both endpoints.
- A response must earn the first point to be eligible for second point.
- To earn the second point a response must correctly discuss the behavior of f', the slopes of f', or the sign of f''.
- Special case: A response that presents exactly one of the two correct intervals with a correct reason earns 1 out of 2 points.

Total for part (b) 2 points

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x \to 2} f(x) = f(2) = 1$. $\lim_{x \to 2} (6f(x) - 3x) = 6 \cdot 1 - 3 \cdot 2 = 0$ $\lim_{x \to 2} \left(x^2 - 5x + 6\right) = 0$	Limits of numerator and denominator	**1 point**
Because $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$ is of indeterminate form $\dfrac{0}{0}$, L'Hospital's Rule can be applied. Using L'Hospital's Rule, $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x \to 2} \dfrac{6f'(x) - 3}{2x - 5} = \dfrac{6 \cdot 0 - 3}{2 \cdot 2 - 5} = 3$	Uses L'Hospital's Rule	**1 point**
	Answer	**1 point**

Scoring notes:

- The first point is earned by the presentation of two separate limits for the numerator and denominator.
- A response that presents a limit explicitly equal to $\dfrac{0}{0}$ does not earn the first point.
- The second point is earned by applying L'Hospital's Rule, that is, by presenting at least one correct derivative in the limit of a ratio of derivatives.
- The third point is earned for the correct answer with supporting work.

Total for part (c) 3 points

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(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

$f'(x) = 0 \Rightarrow x = -1,\ x = 2,\ x = 6$	Considers $f'(x) = 0$	**1 point**
The function f is continuous on $[-2, 8]$, so the candidates for the location of an absolute minimum for f are $x = -2$, $x = -1$, $x = 2$, $x = 6$, and $x = 8$. $\begin{array}{c\|c} x & f(x) \\ \hline -2 & 3 \\ -1 & 4 \\ 2 & 1 \\ 6 & 7 - \pi \\ 8 & 11 - 2\pi \end{array}$ The absolute minimum value of f is $f(2) = 1$.	Justification	**1 point**
	Answer	**1 point**

Scoring notes:

- To earn the first point, a response must state $f' = 0$ or equivalent. Listing the zeros of f' is not sufficient.
- A response that presents any error in evaluating f at any critical point or endpoint will not earn the justification point.
- A response need not present the value of $f(-1)$ provided $x = -1$ is eliminated because it is the location of a local maximum. A response need not present the value of $f(6)$ provided $x = 6$ is eliminated by reference to part (a), or eliminated through analysis.
- A response need not present the value of $f(8)$ provided there is a presentation that argues $f'(x) \geq 0$ for $x > 2$ and therefore $f(8) > f(2)$.
- A response that does not consider both endpoints does not earn the justification point.
- The answer point is earned only for indicating that the minimum value is 1. It is not earned for noting that the minimum occurs at $x = 2$.

Total for part (d) 3 points

Total for question 4 9 points

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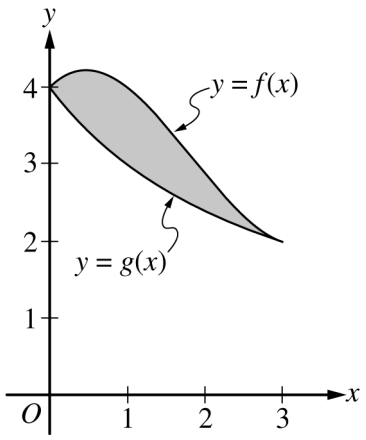

5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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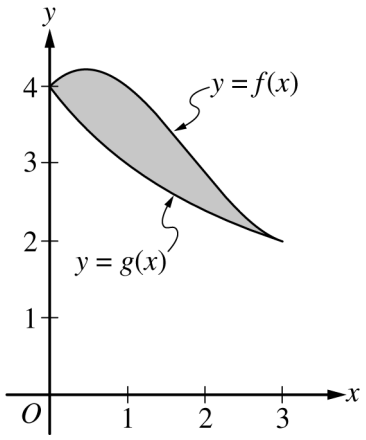

5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

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(b) Evaluate the improper integral $\int_0^{\infty} (g(x))^2 \, dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x) \, dx$.

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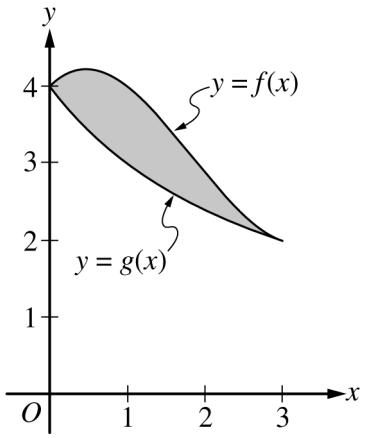

5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

(a)

$$Area = \int_0^3 [f(x) - g(x)]dx = \int_0^3 f(x)dx - \int_0^3 \frac{12}{3+x}\,dx$$
$$= 10 - 12[\ln|3+x|]_0^3 = 10 - 12[\ln 6 - \ln 3] = 10 - 12\ln 2$$

3: {1: integrand; 1: antiderivative; 1: answer}

(b)

$$\int_0^\infty \big(g(x)\big)^2 dx = \lim_{b\to\infty} \int_0^b \frac{144}{(3+x)^2}\,dx = 144 \lim_{b\to\infty} [-(3+x)]_0^b$$
$$= -144 \lim_{b\to\infty} \left[\frac{1}{3+b} - \frac{1}{3}\right] = -144\left(0 - \frac{1}{3}\right) = \frac{144}{3}$$

3: {1: Limit notation; 1: antiderivative; 1: answer}

(c)

$$\int_0^3 \underbrace{x}_{u} \cdot \underbrace{f'(x)dx}_{dv} = \underbrace{(xf(x))}_{uv}\Big|_0^3 - \int_0^3 \underbrace{f(x)dx}_{vdu} = 3f(3) - 0 \cdot f(0) - 10$$
$$= 3(2) - 10 = -4$$

3: {1: u and dv; 1: $\int h(x)dx = x \cdot f(x) - \int f(x)dx$; 1: answer}

Model Solution	Scoring	

(a) Find the area of the shaded region enclosed by the graphs of f and g.

$\text{Area} = \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$	Integrand	**1 point**
$= 10 - \int_0^3 \frac{12}{3+x}\,dx = 10 - 12[\ln\lvert 3+x\rvert]_0^3$	Antiderivative of $g(x)$	**1 point**
$= 10 - 12(\ln 6 - \ln 3) = 10 - 12(\ln 2)$	Answer	**1 point**

Scoring notes:

- The first point is earned for any of the integrands $f(x) - g(x)$, $g(x) - f(x)$, $\lvert f(x) - g(x)\rvert$, or $\lvert g(x) - f(x)\rvert$ in any definite integral. If the limits are incorrect, the response does not earn the third point.
- The first point is earned with an implied integrand for f and explicit integrand for g such as $10 - \int_0^3 g(x)\,dx$.
- The second point is earned for finding $a\int \frac{dx}{3+x} = a \cdot \ln\lvert 3+x\rvert$ or $a \cdot \ln(3+x)$.
- A response is eligible for the third point only if it has earned the first 2 points. The third point is earned only for the correct answer. The answer does not need to be simplified; however, if simplification is attempted, it must be correct.
- A response is not eligible for the third point with incorrect limits of integration for u-substitution, for example, $\int_0^3 \frac{12}{3+x}\,dx = \int_0^3 \frac{12}{u}\,du = 12[\ln(x+3)]_0^3$.
- A response with incorrect communication such as "$\text{Area} = \int_0^3 (g(x) - f(x))\,dx = 10 - 12(\ln 2)$" does not earn the third point. However, a response of "$\int_0^3 (g(x) - f(x))\,dx = 12(\ln 2) - 10$, so the area is $10 - 12(\ln 2)$" earns all 3 points.

Total for part (a) 3 points

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(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

$\int_0^\infty (g(x))^2\,dx = \lim_{b\to\infty}\int_0^b \frac{144}{(3+x)^2}\,dx$	Limit notation	**1 point**
$= \lim_{b\to\infty}\left(-\frac{144}{(3+x)}\Big\vert_0^b\right)$	Antiderivative	**1 point**
$= \lim_{b\to\infty}\left(-\frac{144}{3+b}+\frac{144}{3}\right) = 48$	Answer	**1 point**

Scoring notes:

- To earn the first point, a response must correctly use limit notation throughout the problem and not include arithmetic with infinity, for example, $\left[-\frac{144}{3+x}\right]_0^\infty$ or $-\frac{144}{3+\infty}+48$.
- The second point can be earned by finding an antiderivative of the form $-\frac{a}{(3+x)}$ for $a>0$, from an indefinite or improper integral, with or without correct limit notation. If $a \neq 144$, the response does not earn the third point.
- The third point is earned only for an answer of 48 (or equivalent).
- A response is not eligible for the third point with incorrect limits of integration for u-substitution, for example, $\lim_{b\to\infty}\int_0^b \frac{144}{u^2}\,du = \lim_{b\to\infty}\left[-\frac{144}{3+x}\right]_0^b$.

Total for part (b) 3 points

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(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Using integration by parts, $u = x \quad dv = f'(x)\,dx$ $du = dx \quad v = f(x)$	u and dv	**1 point**
$\int h(x)\,dx = \int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$	$\int h(x)\,dx$ $= x \cdot f(x) - \int f(x)\,dx$	**1 point**
$\int_0^3 h(x)\,dx = \int_0^3 x \cdot f'(x)\,dx = x \cdot f(x)\Big\|_0^3 - \int_0^3 f(x)\,dx$ $= (3 \cdot f(3) - 0 \cdot f(0)) - 10 = 3 \cdot 2 - 0 - 10 = -4$	Answer	**1 point**

Scoring notes:

- The first and second points are earned with an implied u and dv in the presence of $x \cdot f(x) - \int f(x)\,dx$ or $x \cdot f(x)\Big|_0^3 - 10$.
- Limits of integration may be present, omitted, or partially present in the work for the first and second points.
- The tabular method may be used to show integration by parts. In this case, the first point is earned by having columns (labeled or unlabeled) that begin with x and $f'(x)$. The second point is earned for $x \cdot f(x) - \int f(x)\,dx$.
- The third point is earned only for the correct answer and can only be earned if the first 2 points were earned.

Total for part (c) 3 points

Total for question 5 9 points

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6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

   (a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

   (b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

   (c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

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6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

   (a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

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(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

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(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

(a)

$$f^{(4)}(x) = \frac{d}{dx}f'''(x) = -2f'(x^2) + (-2x)(2x)f''(x^2)$$
$$= -2f'(x^2) - 4x^2\left(-f(x^2)\right)$$
$$T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 2 + 3x + \frac{-f(0)}{2!}x^2 + \frac{-2 \cdot 0 \cdot f'(0)}{3!}x^3 + \frac{-2f'(0) - 4 \cdot 0^2\left(-f(0)\right)}{4!}x^4$$
$$= 2 + 3x - \frac{2}{2}x^2 + \frac{0}{6}x^3 - \frac{2(3) - 0}{24}x^4 = 2 + 3x - x^2 - \frac{6}{24}x^4$$

4: 1: Product or chain rule; 1: $f^{(4)}(x)$; 1: two terms; 1: remaing terms

(b)

$$|f(0.1) - T_4(0.1)| \le \frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|x^5}{5!} \le \frac{15 \cdot 0.1^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{8 \cdot 10^5}$$
$$\le \frac{1}{10^5}$$

2: 1: Form of error bound; 1: analysis

(c) $g'(0) = e^0 f(0) = 1 \cdot 2 = 2$
$g''(x) = e^x f(x) + e^x f'(x)$
$g''(0) = e^0 f(0) + e^0 f'(0) = 1 \cdot 2 + 1 \cdot 3 = 5$
$T_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 = 4 + 2x + \frac{5}{2}x^2$

3: 1: $g''(x)$; 1: First two terms; 1: Taylor polynomial

Part A (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

Model Solution	Scoring	
(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.		
$f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right) \cdot 2x$	Form of product rule	**1 point**
	$f^{(4)}(x)$	**1 point**
$f''(0) = -f(0) = -2$ $f'''(0) = -2(0) \cdot f'(0) = 0$ $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$ The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4$ $= 2 + 3x - x^2 - \frac{1}{4}x^4$	Two terms of polynomial	**1 point**
	Remaining terms	**1 point**

Scoring notes:

- The first point is earned for a correct fourth derivative, or for $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right)$.
- The second point is earned only for a completely correct expression for $f^{(4)}(x)$.
- A response that earns the first point but not the second may evaluate the presented expression for $f^{(4)}(x)$ at $x = 0$ and use the consistent nonzero value in computing the coefficient of x^4 in the fourth degree Taylor polynomial.
- A polynomial that includes a nonzero third-degree term, any terms of degree greater than four, or $+\ldots$ does not earn the fourth point.

Total for part (a) 4 points

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(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

By the Lagrange error bound, $\left\|T_4(0.1) - f(0.1)\right\| \le \frac{\max\limits_{0 \le x \le 0.1}\left\|f^{(5)}(x)\right\|}{5!} \cdot (0.1)^5$ $\le \frac{15}{120} \cdot \frac{1}{10^5} \le \frac{1}{10^5}$	Form of error bound	**1 point**
	Shows $\left\|\text{Error}\right\| \le \frac{1}{10^5}$	**1 point**

Scoring notes:

- The first point is earned for $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$. Subsequent errors in simplification will not earn the second point.
- To earn the second point a response must communicate the inequality $\text{Error} \le \frac{15}{5!} \cdot (0.1)^5 \le \frac{1}{10^5}$.
- A response that states $\text{Error} = \frac{15}{5!} \cdot (0.1)^5$ or $\text{Error} = \frac{1}{10^5}$ does not earn the second point.

Total for part (b) 2 points

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(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

$g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$	$g''(x)$	**1 point**
$g'(0) = e^0 \cdot f(0) = 2$ $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0) = 2 + 3 = 5$ The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$.	First two terms of polynomial	**1 point**
	Taylor polynomial	**1 point**

Scoring notes:

- The first point is earned for $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$, $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0)$, or $g''(0) = f(0) + f'(0)$.
- A presented polynomial of the form $4 + 2x + ax^2$ earns the second point with or without any supporting work for the first two terms.
- A response that earned neither the first nor the second point only earns the third point for a polynomial of the form $a + bx + \frac{c}{2}x^2$, where $c \neq 0$ is declared to be $g''(0)$.
- A presented polynomial with no support for the coefficient of x^2 does not earn the third point.
- A polynomial that includes any terms of degree greater than two, or $+\ldots$, does not earn the third point.
- **Alternate solution**
  - $e^x = 1 + x + \frac{x^2}{2} + \cdots$
  - $e^x f(x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(2 + 3x - x^2 + \cdots\right) = 2 + 5x + \cdots$
  - $g(x) = \int e^x f(x)\,dx = C + 2x + \frac{5}{2}x^2 + \cdots$
  - $g(0) = 4 \Rightarrow C = 4$
  - $g(x) \approx 4 + 2x + \frac{5}{2}x^2$

- A response that is using this alternate solution method earns the first point for $e^x f(x) = 2 + 5x + \cdots$, the second point for any two correct terms in a second-degree polynomial, and the third point for a completely correct second-degree Taylor polynomial with supporting work.
- Note: There is not enough information to conclude that $f(x)$ is equal to its Maclaurin series on its interval of convergence. The second and third lines of the alternate solution are being accepted as identifications of the Maclaurin series for $e^x f(x)$ and $g(x)$, respectively.

Total for part (c) 3 points

Total for question 6 9 points

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t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

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t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.
   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.
   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.
   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.
   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

(a) $C'(5) \approx \frac{C(7) - C(3)}{7 - 3} = \frac{69 - 85}{4} = -\frac{16}{4} = -4$ ° per minute

2: {1: Estimate with supporting work; 1: Units}

(b) $\int_0^{12} C(t)dt \approx (100)(3) + (85)(4) + (69)(5) = 985$

$\frac{1}{12}\int_0^{12} C(t)dt$ is the average temperature of the coffee on the interval $0 \le t \le 12$ minutes.

3: {1: Form of left Riemann sum; 1: estimate; 1: interpretation}

(c) $C(20) = 55 + \int_{12}^{20} C'(t)dt = 40.3291\cdots$

3: {1: integral; 1: Uses initial condition; 1: Answer}

(d) $C(t)$ is changing at an increasing rate for $12 < t < 20$ because $C''(t) > 0$ on the interval $12 < t < 20$.

1: Answer with reason

Part A (AB or BC): Graphing calculator required
Question 1 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

	Model Solution	Scoring	
(a)	Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.		
	$C'(5) \approx \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4} = -4$ degrees Celsius per minute	Estimate with supporting work	**1 point**
		Units	**1 point**

Scoring notes:

- To earn the first point a response must include a difference and a quotient as the supporting work.
- $\frac{-16}{7-3}$, $\frac{69-85}{7-3}$, or $\frac{69-85}{4}$ is sufficient to earn the first point.
- A response that presents only units without a numerical approximation for $C'(5)$ does not earn the second point.
- The second point is also earned for "degrees per minute" attached to a numerical value.

Total for part (a) 2 points

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(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

$\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7)$ $= 3\cdot 100 + 4\cdot 85 + 5\cdot 69 = 985$	Form of left Riemann sum	**1 point**
	Estimate	**1 point**
$\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval from $t = 0$ to $t = 12$.	Interpretation	**1 point**

Scoring notes:

- Read "$=$" as "$\approx$" for the first point.
- To earn the first point at least five of the six factors in the Riemann sum must be correct. If any of the six factors is incorrect, the response does not earn the second point.
- A response of $(3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7)$ earns the first point. Values must be pulled from the table to earn the second point.
- A response of $3\cdot 100 + 4\cdot 85 + 5\cdot 69$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A completely correct right Riemann sum (e.g., $3\cdot 85 + 4\cdot 69 + 5\cdot 55$) earns 1 of the first 2 points. An unsupported answer of 806 does not earn either of the first 2 points.
- Units will not affect scoring for the second point.
- To earn the third point the interpretation must include both "average temperature" and the time interval. The response need not include a reference to units. However, if incorrect units are given in the interpretation, the response does not earn the third point.

Total for part (b) 3 points

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

$C(20) = C(12) + \int_{12}^{20} C'(t)\,dt$	Integral	**1 point**
	Uses initial condition	**1 point**
$= 55 - 14.670812 = 40.329188$ The temperature of the coffee at time $t = 20$ is 40.329 degrees Celsius.	Answer	**1 point**

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Scoring notes:

- The first point is earned for a definite integral with integrand $C'(t)$. If the limits of integration are incorrect, the response does not earn the third point.
- A linkage error such as $C(20) = \int_{12}^{20} C'(t)\,dt = 55 - 14.670812$ or $\int_{12}^{20} C'(t)\,dt = -14.670812 = 40.329188$ earns the first 2 points but does not earn the third point.
- Missing differential (dt):
  - Unambiguous responses of $C(20) = C(12) + \int_{12}^{20} C'(t)$ or $C(20) = 55 + \int_{12}^{20} C'(t)$ earn the first 2 points and are eligible for the third point.
  - Ambiguous responses of $C(20) = \int_{12}^{20} C'(t) + C(12)$ or $C(20) = \int_{12}^{20} C'(t) + 55$ do not earn the first point, earn the second point, and earn the third point if the given numeric answer is correct. If there is no numeric answer given, these responses do not earn the third point.
- The second point is earned for adding $C(12)$ or 55 to a definite integral with a lower limit of 12, either symbolically or numerically.
- The third point is earned for an answer of $55 - 14.671$ or $-14.671 + 55$ with no additional simplification, provided there is some supporting work for these values.
- An answer of just 40.329 with no supporting work does not earn any points.

Total for part (c) 3 points

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Because $C''(t) > 0$ on the interval $12 < t < 20$, the rate of change in the temperature of the coffee, $C'(t)$, is increasing on this interval. That is, on the interval $12 < t < 20$, the temperature of the coffee is changing at an increasing rate.	Answer with reason	**1 point**

Scoring notes:

- This point is earned only for a correct answer with a correct reason that references the sign of the second derivative of C.
- A response that provides a reason based on the evaluation of $C''(t)$ at a single point does not earn this point.
- A response that uses ambiguous pronouns (such as "It is positive, so increasing") does not earn this point.
- A response does not need to reference the interval $12 < t < 20$ to earn the point.

Total for part (d) 1 point

Total for question 1 9 points

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2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

   (a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

   (b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

   (c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

   (d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

   (a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

   (b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

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A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

(d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

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2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

   (a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

   (b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

   (c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

   (d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

(a) Speed at $t = 2 \rightarrow \sqrt{\left(x'(2)\right)^2 + \left(y'(2)\right)^2} = 12.304$ (or 12.305)

2: {1: Sertup for speed; 1: answer

(b) Total distance $= \int_0^2 \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$
$= 15.901$ (or 15.902)

2: {1: integral; 1: answer

(c) $y(2) - y(0) = \int_0^2 y'(t)dt \rightarrow y(0) = y(2) - \int_0^2 y'(t)dt$
$= -1.173$ (or -1.174)

3: {1: Definite integral; 1: uses $y(2)$; 1: answer

(d) Moving toward the x-axis when $y(t) > 0$ and $y'(t) < 0$. In the first quadrant means $y(t) > 0$. $y'(t) < 0$ on the interval $(5.221, 8]$ (or $(5.222, 8]$).

2: {1: Considers sign of $y'(t)$; 1: Answer with reason

Part A (BC): Graphing calculator required
Question 2 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

	Model Solution	Scoring	
(a)	Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.		
	$\sqrt{(x'(2))^2 + (y'(2))^2}$	Setup for speed	**1 point**
	$= 12.3048506$ The speed of the particle at time $t = 2$ seconds is 12.305 (or 12.304) centimeters per second.	Answer	**1 point**

Scoring notes:

- The first point is earned for the expression $\sqrt{(x'(2))^2 + (y'(2))^2}$, $\sqrt{(x'(t))^2 + (y'(t))^2}$, or equivalent.
- A response that presents just the exact answer, $\sqrt{144 + \left(-2 + \sqrt{2^{1.2} + 20}\right)^2}$, earns both points.
- The second point is earned only for the answer 12.305 (or 12.304) regardless of whether the first point is earned.
- A response that includes a linkage error, such as $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{144 + \left(-2 + \sqrt{2^{1.2} + 20}\right)^2}$ or $\sqrt{(x'(t))^2 + (y'(t))^2} = 12.305$ (or 12.304), earns at most 1 of the 2 points.
- Missing or incorrect units will not affect scoring in this part.

Total for part (a) 2 points

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(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$	Integral	**1 point**
$= 15.901715$ The total distance traveled by the particle over the time interval $0 \le t \le 2$ is 15.902 (or 15.901) centimeters.	Answer	**1 point**

Scoring notes:

- The first point is earned only for an integral of $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$ (or the mathematical equivalent), with or without the differential.
  - Note: $\int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\ dt$ is not read as a parenthesis error.
- The second point is earned only for an answer of 15.902 (or 15.901), regardless of whether the first point is earned.
- Missing or incorrect units will not affect scoring in this part.

Total for part (b) 2 points

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

$y(0) = 6 + \int_2^0 y'(t)\ dt = 6 - 7.173613 = -1.173613$	Definite integral	**1 point**
	Uses initial condition	**1 point**
The y-coordinate of the position of the particle at time $t = 0$ is -1.174 (or -1.173).	Answer	**1 point**

Scoring notes:

- An answer of -1.174 (or -1.173) with no supporting work does not earn any points.
- The first point is earned for either of the definite integrals $\int_2^0 y'(t)\ dt$ or $\int_0^2 y'(t)\ dt$.
- The second point is earned for any of:
  - $y(2) \pm \int_2^0 y'(t)\ dt,\ 6 \pm \int_2^0 y'(t)\ dt,$
  - $y(2) \pm \int_0^2 y'(t)\ dt,\ 6 \pm \int_0^2 y'(t)\ dt,$
  - $y(2) \pm 7.173613,$ or $6 \pm 7.173613.$
- A response that attempts to evaluate $\int y'(t)\ dt$ does not earn the first or the third point.
  - Such a response can earn the second point by attempting to solve for the constant of integration by presenting an expression as an antiderivative for $y'(t)$, evaluating this expression at $t = 2$, and setting this expression equal to 6.

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- A response that reverses the limits of integration, e.g., $y(0) = 6 + \int_0^2 y'(t)\,dt$ or $6 + 7.173613$, earns the second point but does not earn the third point.
- In order to earn the third point, a response must have earned at least 1 of the first 2 points.
- A response containing any linkage error can earn at most 2 of the 3 points. For example:
  - Equating two unequal quantities: $\int_2^0 y'(t)\,dt = -1.174$, $\int_2^0 y'(t)\,dt = 6 - 7.173613$, $6 + \int_0^2 y'(t)\,dt = 6 - 7.173613$, or $6 + \int_0^2 y'(t)\,dt = 6 + 7.173613 = -1.173613$
  - Equating an expression to a numerical value: $y(t) = 6 + \int_2^0 y'(t)\,dt = -1.174$
- Missing differentials (dt):
  - Unambiguous responses of $y(2) + \int_2^0 y'(t)$, $y(2) - \int_0^2 y'(t)$, $6 + \int_2^0 y'(t)$, or $6 - \int_0^2 y'(t)$ earn the first 2 points and would earn the third point for the correct numerical answer.
  - Unambiguous responses of $y(2) + \int_0^2 y'(t)$ or $6 + \int_0^2 y'(t)$ with reversed limits of integration and missing differential earn the first 2 points but cannot earn the third point.
  - Ambiguous responses of $\int_2^0 y'(t) + y(2)$, $-\int_0^2 y'(t) + y(2)$, $\int_2^0 y'(t) + 6$, or $-\int_0^2 y'(t) + 6$ earn the first point, do not earn the second point, but do earn the third point if a correct numeric answer is provided. If no numeric answer is given, none of these responses earn the third point.
  - Ambiguous responses of $\int_0^2 y'(t) + y(2)$ or $\int_0^2 y'(t) + 6$ with reversed limits of integration and no differential earn 1 out of 3 points.
- If a response provides work for both the x- and y-coordinates, the work for the x-coordinate will not affect scoring.
- However, a response that reports only a completely correct x-coordinate of the particle's position at time $t = 0$ with all supporting work, e.g., $x(0) = 3 + \int_2^0 x'(t)\,dt = -\frac{31}{3} = -10.333$, earns 2 out of 3 points.

Total for part (c) 3 points

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(d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Because $y(t) > 0$ when $2 \le t \le 8$, the particle will be moving toward the x-axis when $y'(t) < 0$. This occurs when 5.222 (or 5.221) $< t < 8$.	Considers sign of $y'(t)$	**1 point**
	Answer with reason	**1 point**

Scoring notes:

- The first point can be earned by stating $y'(t) = 0$, $y'(t) < 0$, $y'(t) > 0$, or $t = 5.222$ (or 5.221).
  Note: $y'(t)$ may be written as $\frac{dy}{dt}$.
- The second point cannot be earned without the first.
- To earn the second point, a response must identify the correct interval (and no additional intervals in $[2, 8]$) and explicitly state the need for $y'(t) < 0$. The interval can be open, closed, or half-open.

Total for part (d) 2 points

Total for question 2 9 points

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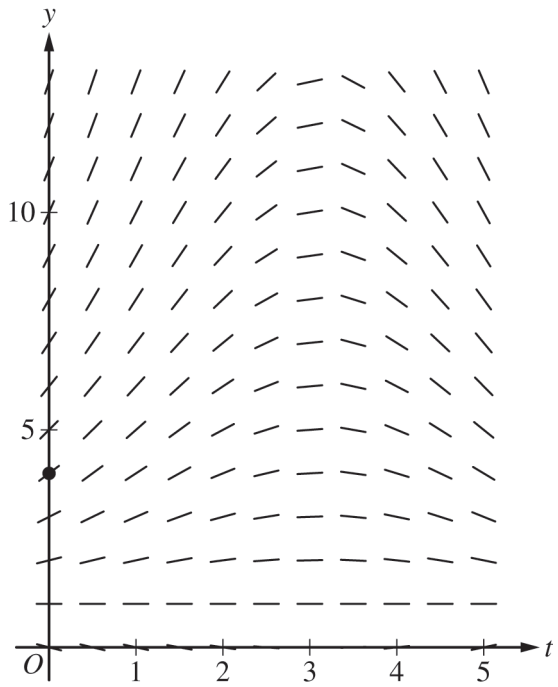
3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

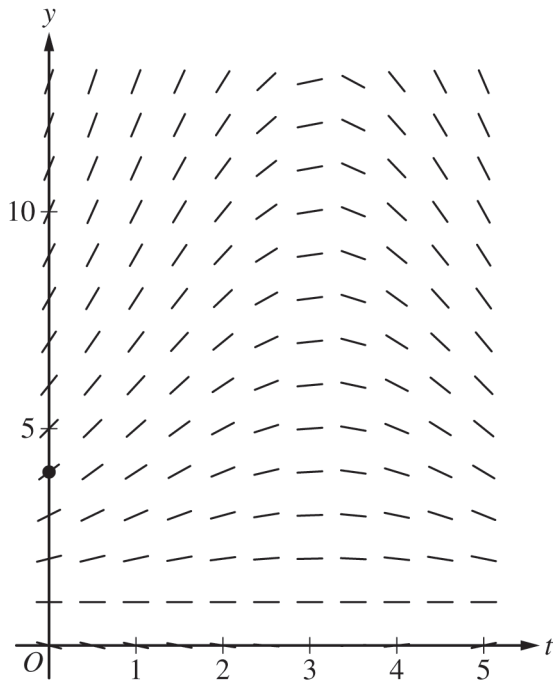
(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

NO CALCULATOR ALLOWED

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

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NO CALCULATOR ALLOWED

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

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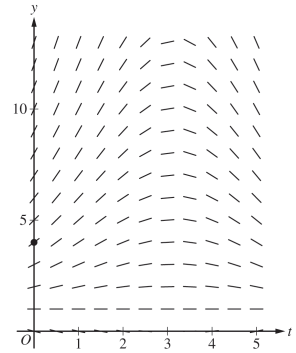
3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

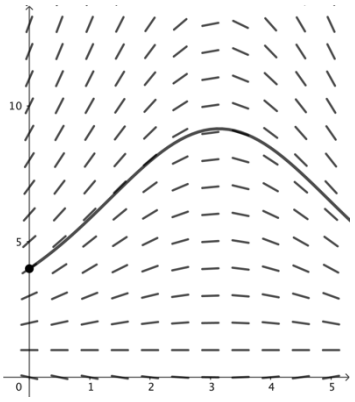


(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0,4)$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

(a)



1: solution curve

(b) $\frac{dH}{dt} = 0 \rightarrow H - 1 = 0 \rightarrow H = 1$ but $H > 1$

and $\frac{1}{2}(H-1) > 0$ for $0 < t < 5$.

$\cos\left(\frac{t}{2}\right) = 0 \rightarrow \frac{t}{2} = \frac{\pi}{2}\cdots \rightarrow t = \pi, 3\pi, \cdots 0 < t = \pi < 5.$

$\cos\left(\frac{t}{2}\right)$ changes from positive to negative at $t = \pi$

so there is a relative maximum at $t = \pi$.

3:
- 1: Considers sign of $\frac{dH}{Dt}$
- 1: $t = \pi$
- 1: answer with justification

(c) $\frac{1}{H-1}dH = \frac{1}{2}\cos\left(\frac{t}{2}\right) \rightarrow \int \frac{1}{H-1}dH = \int \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$

$\ln(H-1) = \sin\left(\frac{t}{2}\right) + C$

$H(0) = 4 \rightarrow \ln 3 = \sin 0 + C \rightarrow C = \ln 3$

$\ln(H-1) = \sin\left(\frac{t}{2}\right) + \ln 3 \rightarrow H - 1 = e^{\sin\left(\frac{t}{2}\right)+\ln 3} = 3e^{\sin\left(\frac{t}{2}\right)}$

$H = 1 + 3e^{\sin\left(\frac{t}{2}\right)}$

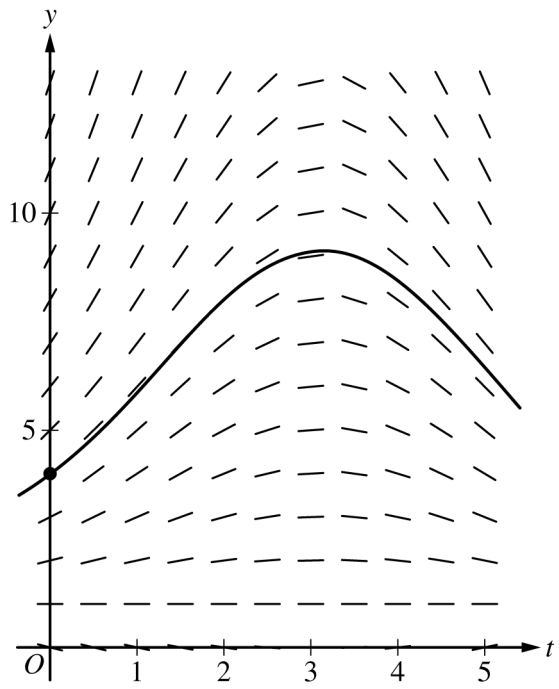
5:
- 1: Separation
- 1: One antiderivative.
- 1: Second antiderivative
- 1: Constant of integration uses initial condition
- 1: Solves for H

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

	Model Solution	Scoring
(a)	A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.	
		Solution curve **1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 4)$, extend to at least $t = 4.5$, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.

Total for part (a) 1 point

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(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Because $H(t) > 1$, then $\frac{dH}{dt} = 0$ implies $\cos\left(\frac{t}{2}\right) = 0$.	Considers sign of $\frac{dH}{dt}$	**1 point**
This implies that $t = \pi$ is a critical point.	Identifies $t = \pi$	**1 point**
For $0 < t < \pi$, $\frac{dH}{dt} > 0$ and for $\pi < t < 5$, $\frac{dH}{dt} < 0$. Therefore, $t = \pi$ is the location of a relative maximum value of H.	Answer with justification	**1 point**

Scoring notes:

- The first point is earned for considering $\frac{dH}{dt} = 0$, $\frac{dH}{dt} > 0$, $\frac{dH}{dt} < 0$, $\cos\left(\frac{t}{2}\right) = 0$, $\cos\left(\frac{t}{2}\right) > 0$, or $\cos\left(\frac{t}{2}\right) < 0$.
- The second point is earned for identifying $t = \pi$, with or without supporting work. A response may consider $H = 1$ or $t = 1$ as potential critical points without penalty.
- The third point cannot be earned without the first point. The third point is earned only for a correct justification and a correct answer of "relative maximum."
- The justification can be shown by determining the sign of $\frac{dH}{dt}$ (or $\cos\left(\frac{t}{2}\right)$) at a single value in $0 < t < \pi$ and at a single value in $\pi < t < 5$. It is not necessary to state that $\frac{dH}{dt}$ does not change sign on these intervals.
- The third point can also be earned by using the Second Derivative Test. For example:

  $$\frac{d^2H}{dt^2} = \frac{1}{2}(H-1)\left(-\frac{1}{2}\sin\left(\frac{t}{2}\right)\right) + \cos\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dH}{dt}$$

  $$\left.\frac{d^2H}{dt^2}\right|_{t=\pi} < 0$$

  Therefore, $t = \pi$ is the location of a relative maximum value of H.

Total for part (b) 3 points

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(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

$\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$	Separation of variables	**1 point**
$\int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$ $\Rightarrow \ln\lvert H-1\rvert = \sin\left(\frac{t}{2}\right) + C$	One antiderivative Second antiderivative	**1 point** **1 point**
$\ln\lvert 4-1\rvert = \sin\left(\frac{0}{2}\right) + C \Rightarrow C = \ln 3$ Because $H(0) = 4$, $H > 1$, so $\lvert H-1\rvert = H-1$. $\ln(H-1) = \sin\left(\frac{t}{2}\right) + \ln 3$	Constant of integration and uses initial condition	**1 point**
$H-1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$ $H(t) = 1 + 3e^{\sin(t/2)}$	Solves for H	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points.
- A response that presents $\int \frac{dH}{H-1} = \ln(H-1)$ without absolute value symbols earns that antiderivative point.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 0 for t and 4 for H.
- A response is eligible for the fifth point only if it has earned the first 4 points.
- A response earns the fifth point only for an answer of $H(t) = 1 + 3e^{\sin(t/2)}$ or a mathematically equivalent expression for $H(t)$ such as $H(t) = 1 + e^{\sin(t/2)+\ln 3}$.
- A response does not need to argue that $\lvert H-1\rvert = H-1$ in order to earn the fifth point.
- Special case: A response that presents an incorrect separation of variables of $\frac{1}{2}\cdot\frac{dH}{H-1} = \cos\left(\frac{t}{2}\right)dt$ does not earn the first point or the fifth point, but is eligible for the 2 antiderivative points. If the response earns at least 1 of the 2 antiderivative points, then the response is eligible for the fourth point.

Total for part (c) 5 points

Total for question 3 9 points

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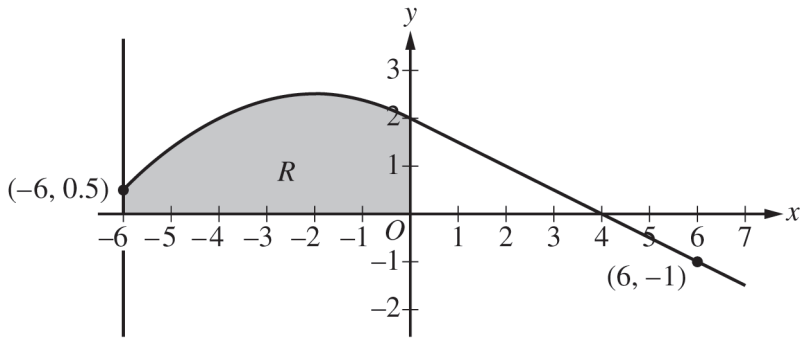


Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

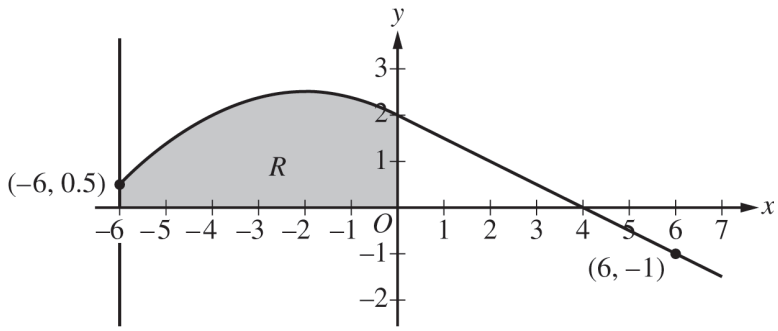
   (a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

   (b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

   (c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

NO CALCULATOR ALLOWED



Graph of f

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(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

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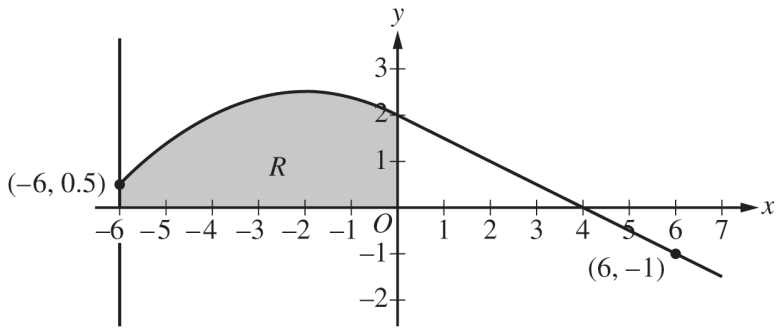

Graph of f

(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

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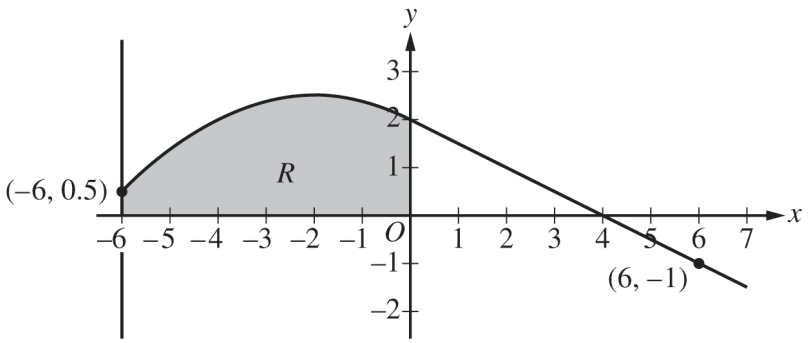

Graph of f

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   (a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

   (b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

   (c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

(a) $g(-6) = \int_0^{-6} f(t)dt = -\int_{-6}^0 f(t)dt = -12$

$g(4) = \int_0^4 f(t)dt = \frac{1}{2}(2)(4) = 4$

$g(6) = g(4) + \int_4^6 f(t)dt = 4 + \frac{1}{2}(-1)(2) = 3$

3: $\begin{cases} 1: g(-6) \\ 1: g(4) \\ 1: g(6) \end{cases}$

(b) $g'(x) = f(x) = 0 \rightarrow x = 4$

2: $\begin{cases} 1: \text{Uses Fundamental Theorem} \\ 1: \text{Answer with reason} \end{cases}$

(c) $h(6) = \int_{-6}^6 f'^{(t)}dt = f(6) - f(-6) = -1 - 0.5 = -1.5$

$h'(6) = f'(6) = -\frac{1}{2}$

$h''(6) = f''(6) = 0$ because f is linear on $(0,7)$ and $h' = f'$ is a constant.

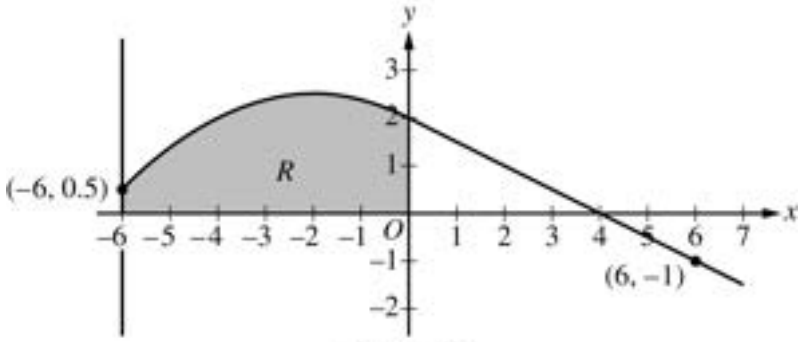
4: $\begin{cases} 1: \text{Uses Fundamental Theorem} \\ 1: h(6) \text{ with supporting work} \\ 1: h'(6) \\ 1: h''(6) \end{cases}$

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f

The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

	Model Solution	Scoring	
(a)	The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.		
	$g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^0 f(t)\,dt = -12$	$g(-6)$	**1 point**
	$g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}\cdot 4\cdot 2 = 4$	$g(4)$	**1 point**
	$g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}\cdot 4\cdot 2 - \frac{1}{2}\cdot 2\cdot 1 = 3$	$g(6)$	**1 point**

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Scoring notes:

- Supporting work is not required for any of these values. However, any supporting work that is shown must be correct to earn the corresponding point.
- Special case: A response that explicitly presents $g(x) = \int_{-6}^{x} f(t)\,dt$ does not earn the first point it would have otherwise earned. The response is eligible for all subsequent points for correct answers, or for consistent answers with supporting work.
  - Note: $\int_{-6}^{-6} f(t)\,dt = 0,\ \int_{-6}^{4} f(t)\,dt = 16,\ \int_{-6}^{6} f(t)\,dt = 15$
- Labeled values may be presented in any order. Unlabeled values are read from left to right and from top to bottom as $g(-6),\ g(4),$ and $g(6),$ respectively. A response that presents only 1 or 2 values must label them to earn any points.

Total for part (a) 3 points

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

$g'(x) = f(x)$	Fundamental Theorem of Calculus **1 point**
$g'(x) = f(x) = 0 \Rightarrow x = 4$ Therefore, the graph of g has a critical point at $x = 4$.	Answer with reason **1 point**

Scoring notes:

- The first point is earned for explicitly making the connection $g' = f$ in this part.
  - A response that writes $g'' = f'$ earns the first point but can only earn the second point by reasoning from $f = 0$.
- A response that does not earn the first point is eligible to earn the second point with an implied application of the FTC (e.g., "Because $g'(4) = 0,\ x = 4$ is a critical point").
- A response that reports any additional critical points in $0 < x < 6$ does not earn the second point.
  - Any presented critical point outside the interval $0 < x < 6$ will not affect scoring.

Total for part (b) 2 points

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(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

$h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$	Uses Fundamental Theorem of Calculus	**1 point**
	$h(6)$ with supporting work	**1 point**
$h'(x) = f'(x)$, so $h'(6) = f'(6) = -\frac{1}{2}$.	$h'(6)$	**1 point**
$h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$.	$h''(6)$	**1 point**

Scoring notes:

- Labeled values may be presented in any order.
- Unlabeled values are read from left to right and from top to bottom as $h(6)$, $h'(6)$, and $h''(6)$, respectively. A response that presents only 1 or 2 values must label them in order to earn any points.
- A response of $h(6) = -1.5$ does not earn either of the first 2 points. A response of $h(6) = f(6) - f(-6)$ earns the first point but not yet the second point.
- A response of $h(6) = -1 - 0.5$ is the minimum work required to earn both of the first 2 points.
- To earn the third point a response must state either $h'(x) = f'(x)$ or $h'(6) = f'(6)$, and provide an answer of $-\frac{1}{2}$.
- The fourth point is earned for a response of $h''(6) = 0$, with or without supporting work.
- A response that has one or more linkage errors does not earn the first point it would have otherwise earned. For example, $h'(x) = f'(6) = -\frac{1}{2}$ does not earn the third point but is eligible for the fourth point even in the presence of another linkage error, such as $h''(x) = f''(6) = 0$.

Total for part (c) 4 points

Total for question 4 9 points

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x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

   (a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

   (b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

   (c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

   (d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

NO CALCULATOR ALLOWED

x	0	π	2π
$f'(x)$	5	6	0

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   (a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

   (b) What information does $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

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NO CALCULATOR ALLOWED

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t+5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

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x	0	π	2π
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   (a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

   (b) What information does $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

   (c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

   (d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

(a) $h'(x) = \sqrt{1 + \left(f'(x)\right)^2}$

$h'(\pi) = \sqrt{1 + \left(f'(\pi)\right)^2} = \sqrt{1 + (6)^2} = \sqrt{37}$

2: {1: Fundamental Theorem; 1: answer}

(b) The length of the curve from $x = 0$ to $x = \pi$.

2: {1: length of curve; 1: interval}

(c) $f(0) = 0 \qquad \Delta x = \pi$

$f(\pi) \approx f(0) + f'(0) \cdot \Delta x = 0 + 5\pi$

$f(2\pi) \approx f(\pi) + f'(\pi) \cdot \Delta x = 5\pi + 6\pi = 11\pi$

2: {1: Euler's Method with two steps; 1: approximation}

(d) $u = t + 5 \rightarrow du = dt \qquad dv = \cos\left(\frac{t}{4}\right) \rightarrow v = 4\sin\left(\frac{t}{4}\right)$

$\int (t + 5)\cos\left(\frac{t}{4}\right) dt = (t + 5) \cdot 4\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt$

$= 4(t + 5)\sin\left(\frac{t}{4}\right) - \left(-16\cos\left(\frac{t}{4}\right)\right) + C$

$= 4(t + 5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$

3: {1: u and dv; 1: $uv - \int vdu$; 1: Answer}

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

x	0	π	2π
$f'(x)$	5	6	0

The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

	Model Solution	Scoring	
(a)	For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.		
	$h'(x) = \sqrt{1 + (f'(x))^2}$	Fundamental Theorem of Calculus	**1 point**
	$h'(\pi) = \sqrt{1 + (f'(\pi))^2} = \sqrt{1 + 6^2} = \sqrt{37}$	Answer	**1 point**

Scoring notes:

- A response of $\sqrt{1 + (f'(\pi))^2}$ earns the first point.
- A response of $\sqrt{1 + 6^2}$ alone earns both points.
- A response such as $h'(x) = \sqrt{1 + (f'(x))^2} = \sqrt{37}$, that equates a variable expression to a numeric value, earns at most 1 of the 2 points.
- A response that equates $h'(x)$ or $h'(\pi)$ to a derivative of a constant, such as $h'(x) = \frac{d}{dx}\int_0^{\pi} \sqrt{1 + (f'(t))^2}\, dt$, earns at most 1 of the 2 points.

Total for part (a) 2 points

(b) What information does $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

$\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the arc length of the graph of f on $[0, \pi]$.	Arc length of f	**1 point**
	Interval $[0, \pi]$	**1 point**

Scoring notes:

- A response of "arc length" or "length" earns the first point. Such a response does not need to reference f. However, if the response references a different function, the response does not earn the first point and is eligible to earn the second point.
- A response referring to distance explicitly connected to the graph or f (or equivalent) earns the first point. For example, a response of "distance along the curve" or "distance traveled by a particle moving along f " earns the first point and is eligible to earn the second point.
- A response referring to distance that is not explicitly connected to the graph of f does not earn the first point but is eligible to earn the second point. For example, a response of "distance" or "distance traveled" does not earn the first point but is eligible to earn the second point.
- To earn the second point a response must connect the interval $[0, \pi]$ to arc length, length, or distance.

Total for part (b) 2 points

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

$f(\pi) \approx f(0) + \pi f'(0) = 0 + 5\pi = 5\pi$ $f(2\pi) \approx f(\pi) + \pi f'(\pi)$	Euler's method	**1 point**
$\approx 5\pi + 6\pi = 11\pi$	Answer	**1 point**

Scoring notes:

- To earn the first point a response must demonstrate two Euler's steps, with use of the correct expression for $\frac{dy}{dx}$, and at most one error. If there is an error, the second point is not earned.
- In order to earn the first point, a response that presents a single error in computing the approximation of $f(\pi)$ must import the incorrect value in computing the approximation of $f(2\pi)$.
- The two Euler's steps may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot \pi$)
0	0	5π
π	5π	6π
2π	11π	

  - In the presence of a correct answer, a table does not need to be labeled in order to earn both points. In the presence of no answer or an incorrect answer, such a table must be correctly labeled in order to earn the first point.
- Both points are earned for $5\pi + 6\pi$.
- The response may report the final answer as $(2\pi, 11\pi)$.

Total for part (c) 2 points

(d) Find $\int (t+5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

$u = t+5 \qquad dv = \cos\left(\frac{t}{4}\right) dt$ $du = dt \qquad v = 4\sin\left(\frac{t}{4}\right)$	u and dv	**1 point**
$\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt$	$uv - \int v\, du$	**1 point**
$= 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$	Answer	**1 point**

Scoring notes:

- The first and second points are earned with an implied u and dv in the presence of $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt$ or a mathematically equivalent expression.
- The tabular method may be used to show integration by parts. In this case, the first point is earned by columns (labeled or unlabeled) that begin with $t+5$ and $\cos\left(\frac{t}{4}\right)$. The second point is earned for $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt$ or a mathematically equivalent expression.
- The third point is earned only for an expression mathematically equivalent to $4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$ (such as $4t\sin\left(\frac{t}{4}\right) + 20\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$) in the presence of correct supporting work.
- To earn the third point a response must have a final answer that includes a constant of integration.
- Alternate solution:

$$\int (t+5)\cos\left(\frac{t}{4}\right) dt = \int t\cos\left(\frac{t}{4}\right) dt + \int 5\cos\left(\frac{t}{4}\right) dt$$

$$u = t \qquad dv = \cos\left(\frac{t}{4}\right) dt$$

$$du = dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

$$\int t\cos\left(\frac{t}{4}\right) dt + \int 5\cos\left(\frac{t}{4}\right) dt = 4t\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt + \int 5\cos\left(\frac{t}{4}\right) dt$$

$$= 4t\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + 20\sin\left(\frac{t}{4}\right) + C$$

  - A response can earn the first and second points for correctly applying integration by parts to $\int t\cos\left(\frac{t}{4}\right) dt$. The tabular method may be used to show integration by parts. The third point is earned for the correct answer.

Total for part (d) 3 points

Total for question 5 9 points

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6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

   (a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

   (b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

   (c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

   (d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

NO CALCULATOR ALLOWED

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

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NO CALCULATOR ALLOWED

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

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(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

(a) $f(6) = \sum_{n=1}^{\infty} \frac{(n+1)\cdot 6^n}{n^2 \cdot 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{(n+1)}{n^2}} = \lim_{n\to\infty} \frac{1}{\frac{(n+1)}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series so by the limit comparison test $\sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$ will also diverge. The series for f diverges at $x = 6$

2: {1: Considers $\frac{(n+1)6^n}{n^2 6^n}$; 1: Answer withreason

(b) The series for $f(-3)$ is an alternating series whose terms decrease in absolute value to 0. The error $|f(-3) - S_3| < |t_4|$.

$\left|\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4\right| = \left|\frac{5}{16}\left(\frac{1}{16}\right)^4\right| = \frac{5}{256} < \frac{1}{50} = \frac{5}{250}$

2: {1: uses 4th term; 1: Verification

(c) $f(x) = \sum_{n=1}^{\infty} \frac{(n+1)\cdot x^n}{n^2 \cdot 6^n}$

$f'(x) = \sum_{n=1}^{\infty} \frac{(n+1)\cdot n \cdot x^{n-1}}{n^2 \cdot 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)\cdot x^{n-1}}{n \cdot 6^n}$

The radius of convergence is the same as the radius of convergence of f.

2: {1: general term of f'; 1: radius of convergence

(d) $\lim_{n\to\infty} \left|\frac{(n+2)\cdot x^{2(n+1)}}{(n+1)^2 \cdot 3^{n+1}} \cdot \frac{n^2 \cdot 3^n}{(n+1)\cdot x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{(n+2)n^2 \cdot x^2}{(n+1)^3 \cdot 3}\right|$

$= \frac{1}{3}x^2 < 1 \to x^2 < 3 \to -\sqrt{3} < x < \sqrt{3}$

The radius of convergence is $\sqrt{3}$.

3: {1: ratio test; 1: Limit; 1: radius of convergence

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

	Model Solution	Scoring	
(a)	Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.		
	At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.	Considers $\frac{(n+1)6^n}{n^2 6^n}$	**1 point**
	Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \geq 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the comparison test.	Answer with reason	**1 point**

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Scoring notes:

- To earn the first point using either the comparison or limit comparison test, a response must consider the term $\frac{(n+1)6^n}{n^2 6^n}$. This could be shown by considering the term $\frac{n+1}{n^2}$, either individually or as part of a sum.
- To earn the second point using the comparison test a response must demonstrate that the terms $\frac{n+1}{n^2}$ are larger than the terms in a divergent series.
  - "$\frac{n+1}{n^2} > \frac{1}{n}$, diverges" earns both points.
  - The response does not need to use the term "comparison test," but the response cannot declare use of an incorrect test.
- Alternate solution (limit comparison test):

  At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.

  Because $\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the limit comparison test.

  - To earn the second point using the limit comparison test, a response must correctly write the limit of the ratio of the terms in the given series to the terms of a divergent series and demonstrate that the limit of this ratio is 1.
  - The response does not need to use the term "limit comparison test," but the response cannot declare use of an incorrect test.

Total for part (a) 2 points

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(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

$f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}\left(-\frac{1}{2}\right)^n$ is an alternating series with terms that decrease in magnitude to 0. By the alternating series error bound, $\sum_{n=1}^{3} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n = -\frac{125}{144}$ approximates $f(-3)$ with error of at most $\left\|\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4\right\| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$. Thus, $\|f(-3) - S_3\| < \frac{1}{50}$.	Uses fourth term **1 point** Verification **1 point**

Scoring notes:

- The first point is earned for correctly using $x = -3$ in the fourth term. (Listing the fourth term as part of a polynomial is not sufficient.) Using $x = -3$ in any term of degree five or higher does not earn this point.
- The expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ earns the first point, but just $\frac{5}{256}$ does not earn the first point.
- A response including the expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ that is subsequently simplified incorrectly earns the first point but not the second.
- To earn the second point the response must state that the series for $f(-3)$ is alternating or that the alternating series error bound is being used.
  - A response of just "Error $\le \frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4 < \frac{1}{50}$" (or any equivalent mathematical expression) earns both points, provided it is accompanied by an indication that the series is alternating.
- A response that declares the error is equal to $\frac{5}{256}$ (or any equivalent form of this value) does not earn the second point.

Total for part (b) 2 points

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(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

The general term of the Maclaurin series for f' is $\frac{(n+1)nx^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n \cdot 6^n}$.	General term	**1 point**
Because the radius of convergence of the Maclaurin series for f is 6, the radius of convergence of the Maclaurin series for f' is also 6.	Radius	**1 point**

Scoring notes:

- A response of $\frac{(n+1)nx^{n-1}}{n^2 6^n}$ earns the first point. Any expression mathematically equivalent to this also earns the first point.
- The response need not simplify $\frac{(n+1)nx^{n-1}}{n^2 6^n}$, but any presented simplification must be correct in order to earn the first point.
- The second point is earned only for a supported answer of 6. The second point can be earned without the first.
- Alternate solution for second point (ratio test):

$$\lim_{n\to\infty}\left|\frac{\frac{(n+2)x^n}{(n+1)6^{n+1}}}{\frac{(n+1)x^{n-1}}{n\cdot 6^n}}\right| = \lim_{n\to\infty}\left|\frac{n(n+2)}{(n+1)^2}\cdot\frac{x}{6}\right| = \frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

  Therefore, the radius of convergence of the Maclaurin series for f' is 6.

- Alternate solution for second point (root test):

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)|x|^{n-1}}{n\cdot 6^n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1/n}\cdot|x|^{-1/n}\cdot\frac{|x|}{6} = 1\cdot 1\cdot\frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

  Therefore, the radius of convergence of the Maclaurin series for f' is 6.

Total for part (c) 2 points

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(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \frac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}} \right\rvert = \left\lvert \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}} \right\rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \left\lvert \frac{(n+2)n^2}{(n+1)^3} \right\rvert \cdot \left\lvert \frac{x^2}{3} \right\rvert = \left\lvert \frac{x^2}{3} \right\rvert$	Limit	**1 point**
$\left\lvert \frac{x^2}{3} \right\rvert < 1 \Rightarrow x^2 < 3 \Rightarrow \lvert x \rvert < \sqrt{3}$ The radius of convergence of g is $\sqrt{3}$.	Radius of convergence	**1 point**

Scoring notes:

- The first point is earned by presenting a correct ratio with or without absolute values. Once earned, this point cannot be lost. Any errors in simplification or evaluation of the limit will not earn the second point.
- The first point is earned for ratios mathematically equivalent to any of the following:
  $\frac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}}$, $\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}$, $\frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}}$, or $\frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n-1)^2 3^{n-1}}{nx^{2n-2}}$.
- The first point is also earned for ratios mathematically equivalent to the following reciprocal ratios:
  $\frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}$, $\frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n+1)^2 3^{n+1}}{(n+2)x^{2n+2}}$, $\frac{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}}$, or $\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}$.

  Responses including any of these reciprocal ratios can earn the second point for using limit notation to correctly find a limit of the absolute value of their ratio to be $\left\lvert \frac{3}{x^2} \right\rvert$. Such responses earn the third point only for a final answer of $\sqrt{3}$ with a valid explanation for reporting the reciprocal of $\frac{1}{\sqrt{3}}$.
- To earn the second point a response must use the ratio and correctly evaluate the limit of the ratio, using correct limit notation.
- The third point is earned only for an answer of $\sqrt{3}$ with supporting work.

Total for part (d) 3 points

Total for question 6 9 points

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1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

Part A (AB or BC): Graphing calculator required
Question 1 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

	Model Solution	Scoring	
A	Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.		
	$\frac{1}{4-0}\int_0^4 C(t)\,dt$	Average value formula	**Point 1 (P1)**
	$= \frac{1}{4}(11.112896) = 2.778224$ From time $t = 0$ to $t = 4$ weeks, the average number of acres affected by the invasive species was 2.778 acres.	Answer	**Point 2 (P2)**

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Scoring Notes for Part A

- **P1** is earned for the correct integral, with or without the differential, along with evidence of division by 4. In the presence of the correct integral, the correct answer will suffice as evidence of division by 4. These may appear all in one step, as in the model solution, or in multiple steps.
- **P2** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
- Incorrect or unclear communication between the correct integral and the correct answer is treated as scratch work and is not considered in scoring. For example:
  - $\int_0^4 C(t)\,dt = 11.112896$ so the average velocity is 2.778224.

    Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer.
  - $\int_0^4 C(t)\,dt = \frac{11.112896}{4} = 2.778224$

    Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
  - $\int_0^4 C(t)\,dt = 2.778224$

    Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
- Note that the values $\frac{1}{4}(11.112)$ and $\frac{1}{4}(11.113)$ are accurate to three digits after the decimal and therefore earn **P2**.

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B Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

$\frac{C(4)-C(0)}{4-0} = 1.282008$	Uses average rate of change	**Point 3 (P3)**
$C'(t) = \frac{38}{25+t^2} = 1.282008 \Rightarrow t = 2.154298$ The instantaneous rate of change of C equals the average rate of change of C over the interval $0 \le t \le 4$ at time $t = 2.154$.	Answer with supporting work	**Point 4 (P4)**

Scoring Notes for Part B

- **P3** may be earned by presenting the expression or value for the average rate of change. Note that because $C(0) = 0$ and the interval is $0 \le t \le 4$, any of the following will earn **P3**: $\frac{\int_0^4 C'(t)\,dt}{4}$, $\frac{C(4)-C(0)}{4-0}$, $\frac{C(4)}{4}$, $\frac{5.128-0}{4-0}$, $\frac{5.128}{4}$, or 1.282. However, neither **P3** nor **P4** is earned by just presenting $t = 1.282$.
- **P4** is earned for the correct answer supported by the appropriate equation. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.
  - The following response, for example, earns both **P3** and **P4**: $C'(t) = \frac{C(4)-C(0)}{4}$ when $t = 2.154$.

C Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

$\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2}$	Limit expression	**Point 5 (P5)**
$= 0$	Value	**Point 6 (P6)**

Scoring Notes for Part C

- **P5** can be earned for either $\lim_{t\to\infty} C'(t)$ or $\lim_{t\to\infty} C(t)$.
- A response that includes $\lim_{t\to\infty} C(t)$ is not eligible to earn **P6**.
- For **P6**, arithmetic with infinity, e.g., $\frac{38}{25+\infty^2} = 0$, will be considered as scratch work and will not be considered in scoring.

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AP Calculus AB 2025 Free-Response Questions Scoring Guides

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6 \arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

(a) $\dfrac{1}{4-0}\displaystyle\int_0^4 C(t)dt = 2.778$ is the average number of acres affected from $t = 0$ to $t = 4$ weeks.

2: {P1: Average value formula; P2: Answer}

(b) Average rate of change $Q = \dfrac{C(4) - C(0)}{4 - 0} = 1.282$

$C'(t) = Q \rightarrow t = 2.154$

2: {P3: Uses average rate of change; P4: Answer with supporting work}

(c) $\lim\limits_{t \to \infty} \dfrac{38}{25 + t^2} = 0$ because the limit becomes the form $\dfrac{\text{constant}}{\infty}$

2: {P5: Limit expression; P6: Value}

(d) $A'(t) = C'(t) - 0.1 \ln(t) = 0 \rightarrow t = 11.441 \ldots = p$

t	$A(t)$
4	5.128...
p	7.316 ...
36	1.743 ...

$A(t)$ attains its maximum value at $t = p = 11.441 \ldots$

3: {P7: Considers $A'(t) = 0$; P8: Justification; P9: Answer with supporting work}

D At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\, dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

$A'(t) = C'(t) - 0.1 \cdot \ln t$ For $4 \le t \le 36$, the maximum value of $A(t)$ occurs when $A'(t) = 0$ or at an endpoint. $A'(t) = C'(t) - 0.1 \cdot \ln t = 0 \Rightarrow C'(t) = 0.1 \cdot \ln t$	Considers $A'(t) = 0$	**Point 7 (P7)**
$\Rightarrow t = 11.441700$ t | $A(t)$ 4 | 5.128031 11.441700 | 7.316978 36 | 1.743056	Justification	**Point 8 (P8)**
Therefore, the number of acres affected by the species is a maximum at time $t = 11.442$ (or 11.441) weeks.	Answer with supporting work	**Point 9 (P9)**

Scoring Notes for Part D

- **P7** is earned for considering $A'(t) = 0$, $C'(t) - 0.1 \cdot \ln t = 0$, or $C'(t) = 0.1 \cdot \ln t$. **P7** is not earned by just presenting $t = 11.441700$.
  - A response that discusses the sign of $A'(t)$ changing or uses the phrase "critical points of A" also earns **P7**.
- To earn **P8** using a candidates test, a response must make a global argument by correctly evaluating $A(t)$ at $t = 4$, $t = 11.441700$, and $t = 36$. The evaluations must be correct to the first digit after the decimal (rounded or truncated).
- Alternate justifications:
  - $A'(t) > 0$ for $4 < t < 11.442$, and $A'(t) < 0$ for $11.442 < t < 36$. Therefore, $t = 11.442$ is the location of the absolute maximum for A on the interval $4 \le t \le 36$.
  - Because $A'(t)$ changes sign from positive to negative at $t = 11.442$ (this might be presented as "$A'(t) > 0$ for $t < 11.442$, and $A'(t) < 0$ for $t > 11.442$"), it is the location of a relative maximum for A. And because $t = 11.442$ is the only critical point of A in the interval $4 \le t \le 36$, it is the location of the absolute maximum for A on the interval.
- A response that presents a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P8** but is eligible for **P9** with the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

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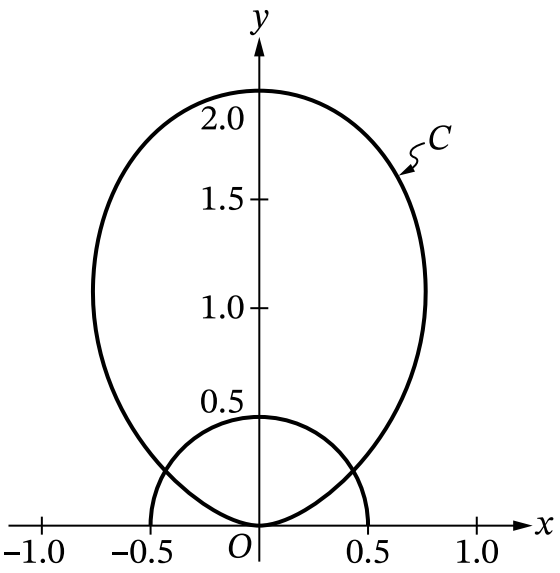
2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

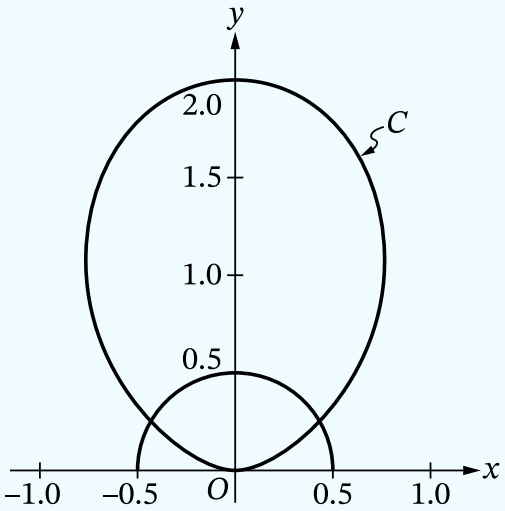
D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

Part A (BC): Graphing calculator required
Question 2 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

	Model Solution	Scoring
A	Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.	
	$\left.\frac{dr}{d\theta}\right\|_{\theta=1.3} = 1.031003$ The rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$ is 1.031.	Answer with setup **Point 1 (P1)**

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Scoring Notes for Part A

- An exact answer of $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3)$ earns **P1**.
- To earn **P1**, a response must indicate differentiation of r and provide the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
  - Examples of responses with correct communication include $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 1.031$ and $r'(1.3) = 1.031$.
  - Responses with incorrect communication, such as $r'(\theta) = 1.031$, $r' = 1.031$, or $\frac{dr}{d\theta} = 1.031$, are sufficient to earn **P1**.

B

Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

For $0 \le \theta \le \pi$, $r(\theta) = \frac{1}{2}$ for $\theta = \theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta = \theta_2 = \frac{5\pi}{6} = 2.617994$.	Integrand including $(r(\theta))^2$ **Point 2 (P2)**
$\frac{1}{2}\int_{\theta_1}^{\theta_2}\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$	Integrand **Point 3 (P3)**
$= 2.066769$	Answer **Point 4 (P4)**
The area is 2.067 (or 2.066).	

Scoring Notes for Part B

- **P2** is earned for a definite integral including $(r(\theta))^2$, such as $\int_a^b\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$ or $\int_a^b (r(\theta))^2\,d\theta$, with or without the differential $d\theta$.
- **P3** is earned for a definite integral (or integrals) with a correct integrand, such as $\int_a^b\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$ or $\int_a^b (r(\theta))^2\,d\theta - \int_c^d \left(\frac{1}{2}\right)^2 d\theta$, with or without the differential $d\theta$.
- The limits $\theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta_2 = \frac{5\pi}{6} = 2.617994$ and the factor $\frac{1}{2}$ are assessed in **P4**, not in **P2** or **P3**.
- **P4** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

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- Incorrect or unclear communication between the correct integral and the correct answer is treated as scratch work and is not considered in scoring. For example:
  - $\int_{\pi/6}^{5\pi/6}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 4.133538$ so the area is 2.067.

    Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer.
  - $\int_{\pi/6}^{5\pi/6}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 2.067$

    Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
- **Special case:** An indefinite integral with a correct integrand does not earn **P2**, earns **P3**, and is eligible to earn **P4** with a correct answer.
- A response of $\int_{\pi/6}^{\pi/2}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 2.067$, using the symmetry of the region, earns **P2**, **P3**, and **P4**.

C

It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

For $0 \le \theta \le \frac{\pi}{2}$, the curve C is in the first quadrant. Thus, a point on the curve will be farthest away from the y-axis when the x-coordinate attains its maximum value. This will either occur when $\frac{dx}{d\theta} = 0$ or at an endpoint of the interval $0 \le \theta \le \frac{\pi}{2}$. $\frac{dx}{d\theta} = 0$	Considers $\frac{dx}{d\theta} = 0$ **Point 5 (P5)**
$\Rightarrow \theta = 0.955317$ $\begin{array}{c\|c} \theta & x(\theta) = r(\theta)\cos\theta \\ \hline 0 & 0 \\ 0.955317 & 0.769800 \\ \frac{\pi}{2} & 0 \end{array}$	Justification **Point 6 (P6)**
Therefore, the value of θ for which the point on the curve is farthest from the y-axis is 0.955.	Answer with supporting work **Point 7 (P7)**

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Scoring Notes for Part C

- **P5** is earned for considering $\frac{dx}{d\theta} = 0$. **P5** is not earned by just presenting $\theta = 0.955317$.
  - A response that discusses the sign of $\frac{dx}{d\theta}$ changing or uses the phrase "critical points of $x(\theta)$" also earns **P5**.
- The value $\theta = 0.955317$ might be presented as $\arccos\left(\frac{1}{\sqrt{3}}\right)$, $\arcsin\left(\sqrt{\frac{2}{3}}\right)$, or $\arctan\left(\sqrt{2}\right)$.
- To earn **P6** using a candidates test, a response must make a global argument by correctly evaluating $x(\theta)$ at $\theta = 0$, $\theta = 0.955317$, and $\theta = \frac{\pi}{2}$. The evaluations must be correct to the first digit after the decimal (rounded or truncated).
- Alternate justifications:
  - $\frac{dx}{d\theta} > 0$ for $0 < \theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $0.955 < \theta < \frac{\pi}{2}$. Therefore, $\theta = 0.955$ is the location of the absolute maximum for $x(\theta)$ on the interval $0 \le \theta \le \frac{\pi}{2}$.
  - Because $\frac{dx}{d\theta}$ changes sign from positive to negative at $\theta = 0.955$ (this might be presented as "$\frac{dx}{d\theta} > 0$ for $\theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $\theta > 0.955$"), it is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \le \theta \le \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
  - Because $\left.\frac{dx}{d\theta}\right|_{\theta=0.955} = 0$ and $\left.\frac{d^2x}{d\theta^2}\right|_{\theta=0.955} < 0$, $\theta = 0.955$ is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \le \theta \le \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
- A response that presents only a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P6** but is eligible for **P7** with the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

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D

A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

$\left.\frac{dr}{dt}\right\|_{\theta=1.3} = \left(\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}\right)\Big\|_{\theta=1.3} = 1.031003\cdot 15 = 15.465041$	$\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$ **Point 8 (P8)**
The particle's distance from the origin changes at a rate of 15.465.	Answer **Point 9 (P9)**

Scoring Notes for Part D

- To earn **P8,** $\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$ can be presented either symbolically or numerically.
- **P8** might be earned in one or more steps.
- A response of $1.031\cdot 15$ or [answer from part A]$\cdot 15$ earns both **P8** and **P9**.
- **P9** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

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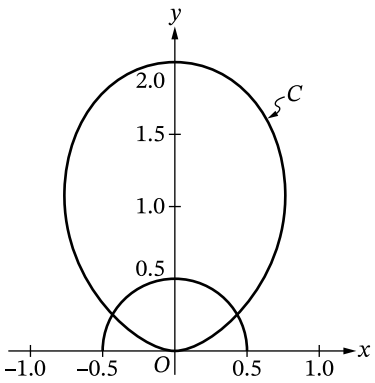
2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

(a) $r'(1.3) = 1.031$

(b) $2\sin^2\theta = \frac{1}{2} \rightarrow \theta = \begin{cases} \alpha = 0.5235\ldots \\ \beta = 2.6179\ldots \end{cases}$

$$\text{area} = \frac{1}{2}\int_{\alpha}^{\beta}\left[\left(r(\theta)\right)^2 - \left(\frac{1}{2}\right)^2\right]d\theta = 2.0667\ldots$$

(c) $\frac{dx}{d\theta} = 0 \rightarrow \theta = 0.8959\ldots = Z$

θ	$x = r(\theta)\cos(\theta)$
0	0
Z	0.7617 ...
$\frac{\pi}{2}$	0

$\theta = 0.8959\ldots$ corresponds to the point on C that is farthest from the y-axis or the x-coordinate is maximum.

(d) $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} \rightarrow \left.\frac{dr}{dt}\right|_{\theta=1.3} = \left.\frac{dr}{d\theta}\right|_{\theta=1.3}\cdot\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = (1.031)(15)$

$= 15.4650\ldots$

P1: answer and set up

3: { P2: Integrand including $\left(r(\theta)\right)^2$; P3: integrand; P4: answer }

3: { P5: Considers $\frac{dx}{d\theta} = 0$; P6: Justification; P7: Answer with supporting work }

2: { P8: $\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$; P9: Answer }

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

	Model Solution	Scoring	
A	Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.		
	$R'(1) \approx \dfrac{R(2) - R(0)}{2 - 0}$ $= \dfrac{100 - 90}{2} = \dfrac{10}{2} = 5$ words per minute per minute	Answer with setup Units	**Point 1 (P1)** **Point 2 (P2)**

Scoring Notes for Part A

- To earn **P1**, a response must present the answer along with the supporting work of a difference and a quotient using values from the table.
  - $\dfrac{100 - 90}{2 - 0}$, $\dfrac{10}{2 - 0}$, $\dfrac{100 - 90}{2}$, or $\dfrac{R(2) - R(0)}{2 - 0} = 5$ is sufficient to earn **P1**.
  - $\dfrac{R(2) - R(0)}{2 - 0}$ by itself is not sufficient to earn **P1**.
- **P2** is earned for correct units, whether or not they are attached to a numerical value for the average rate of change.
- **P2** is also earned for the units "words/minute^2."

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B Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

R is differentiable implies R is continuous.	Differentiable implies continuous	**Point 3 (P3)**
$R(0) = 90 < 155 < R(10) = 162$ Therefore, by the Intermediate Value Theorem, there must be a value c, with $0 < c < 10$, such that $R(c) = 155$.	Answer with justification	**Point 4 (P4)**

Scoring Notes for Part B

- To earn **P3**, a response must state that R is continuous because R is differentiable (or equivalent). A response that simply states "R is continuous" without justification does not earn **P3**.
- A response does not need to earn **P3** to be eligible for **P4**.
- To earn **P4**, a response must indicate that $R(0) < 155$ (or $R(2) < 155$ or $R(8) < 155$) and $R(10) > 155$, state that "R is continuous," and answer "yes" in some way.
- To earn **P4**, a response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

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C Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

$\int_0^{10} R(t)\,dt$ $\approx \frac{R(0)+R(2)}{2}(2-0)+\frac{R(2)+R(8)}{2}(8-2)$ $+\frac{R(8)+R(10)}{2}(10-8)$	Form of trapezoidal sum	**Point 5 (P5)**
$=\frac{90+100}{2}(2-0)+\frac{100+150}{2}(8-2)+\frac{150+162}{2}(10-8)$ $=\frac{190}{2}(2)+\frac{250}{2}(6)+\frac{312}{2}(2)=190+750+312=1252$	Answer with supporting work	**Point 6 (P6)**

Scoring Notes for Part C

- Read "=" as "$\approx$" for **P5**.
- The form of a trapezoidal sum includes three terms, each of which includes a product of two factors, where one of the factors incorporates the $\frac{1}{2}$ as part of the product. To earn **P5**, at least five of the six factors must be correct. If any of the six factors is incorrect, the response does not earn **P6**. Consider the following examples:
  - $\frac{90+100}{2}(2-0)+\frac{100+150}{2}(8-2)+\frac{150+162}{2}(10-8)$ earns **P5** and is sufficient to earn **P6**.
  - $\frac{190}{2}(2)+\frac{250}{2}(6)+\frac{312}{2}(2)$ earns **P5** and is sufficient to earn **P6**.
  - $\frac{1}{2}((R(0)+R(2))(2)+(R(2)+R(8))(6)+(R(8)+R(10))(2))$ earns **P5** and is eligible for **P6**.
  - $\frac{90+100}{2}(2)+\frac{100+150}{2}(2)+\frac{150+162}{2}(2)$ earns **P5** but is not eligible for **P6**.
    (Note that the factor of 2 in the second term of this expression is incorrect.)
- **Special case:** A response of $(90+100)+(100+150)3+(150+162)$ earns both **P5** and **P6**.
- To be eligible for **P6**, a response must have earned **P5**.
  - **Special case:** A response of $95\cdot 2+125\cdot 6+156\cdot 2$ earns **P6** but does not earn **P5**.
- A response of $\frac{90+100}{2}(2-0)+\frac{100+150}{2}(8-2)+\frac{150+162}{2}(10-8)$ or equivalent banks **P6** (i.e., subsequent errors in simplification will not be considered in scoring for **P6)**.
- A response of $\frac{(90\cdot 2+100\cdot 6+150\cdot 2)+(100\cdot 2+150\cdot 6+162\cdot 2)}{2}$ or equivalent earns both **P5** and **P6**. (Note that the average of the left Riemann sum and right Riemann sum is equivalent to the trapezoidal sum.)
- A completely correct left Riemann sum (e.g., $90\cdot 2+100\cdot 6+150\cdot 2=1080$) or a completely correct right Riemann sum (e.g., $100\cdot 2+150\cdot 6+162\cdot 2=1424$) earns **P5** but does not earn **P6**.

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D A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

$\int_0^{10} W(t)\,dt = \int_0^{10}\left(-\frac{3}{10}t^2 + 8t + 100\right)dt$	Integrand	**Point 7 (P7)**
$= \left(-\frac{1}{10}t^3 + 4t^2 + 100t\right)\Big\|_0^{10}$	Antiderivative	**Point 8 (P8)**
$= \left(-\frac{1}{10}\cdot 1000 + 4\cdot 100 + 100\cdot 10\right) - \left(-\frac{1}{10}\cdot 0 + 4\cdot 0 + 100\cdot 0\right)$ $= 1300$ Based on the model, the teacher has read 1300 words by the end of the 10 minutes.	Answer	**Point 9 (P9)**

Scoring Notes for Part D

- **P7** is earned for an indefinite or definite integral with integrand $W(t)$, with or without the differential dt.
- **P8** is earned for the correct antiderivative, with or without the constant of integration.
- To be eligible for **P9**, a response must have earned **P8**.
- A response of $\left(-\frac{1}{10}\cdot 1000 + 4\cdot 100 + 100\cdot 10\right) - \left(-\frac{1}{10}\cdot 0 + 4\cdot 0 + 100\cdot 0\right)$ or equivalent banks **P9** (i.e., subsequent errors in simplification will not be considered in scoring for **P9**).

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3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

(a) $R'(1) \approx \dfrac{R(2) - R(0)}{2 - 0} = \dfrac{100 - 90}{2} = \dfrac{10}{2} = 5$

words per minute per minute.

2: {P1: Answer with set up; P2: Units}

(b) Yes there must be a value c, for $0 < c < 10$, such that $R(c) = 155$. R is differentiable so R is continuous and $R(0) = 90 < 155 < 162 = R(10)$. The Intermediate Value Theorem guarantees there is at least one value c between 8 and 10 where $R(c) = 155$.

2: {P3: Differentiable implies continuous; P4: Answer with justification}

(c) $\displaystyle\int_0^{10} R(t)dt \approx \left(\frac{90 + 100}{2}\right)(2) + \left(\frac{100 + 150}{2}\right)(6) + \left(\frac{150 + 162}{2}\right)(2) = \left(\frac{190}{2}\right)(2) + \left(\frac{250}{2}\right)(6) + \left(\frac{312}{2}\right)(2)$

$= 190 + 750 + 312 = 1252$

2: {P5: Form of Trapezoidal sum; P6: Answer with supporting work}

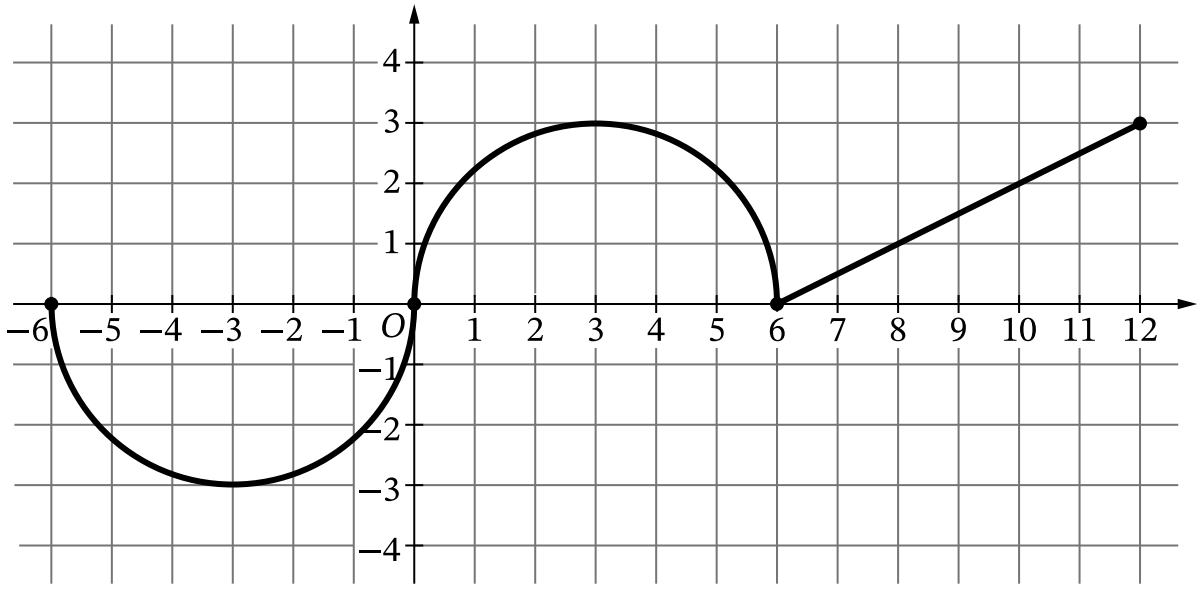
(d) $\displaystyle\int_0^{10} W(t)dt = \left[-\frac{3}{10} \cdot \frac{1}{3}t^3 + 8 \cdot \frac{1}{2}t^2 + 100t\right]_0^{10}$

$\displaystyle= -\frac{1}{10}(10)^3 + 4(10)^2 + 100(10)$

$= -100 + 400 + 1000 = 1300$

3: {P7: Integrand; P8: Antiderivative; P9: Answer}

4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

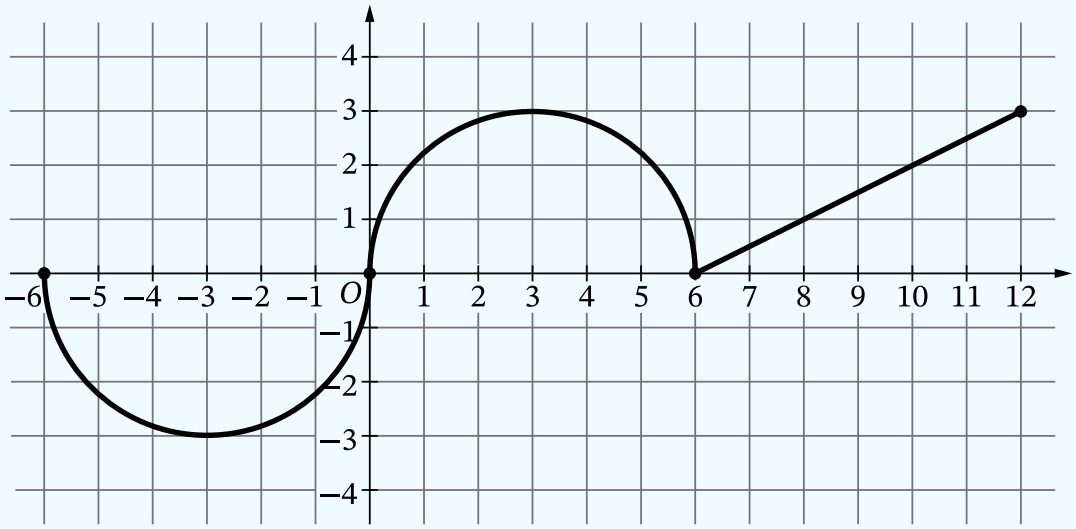
D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

Part A (AB or BC): Graphing calculator not allowed
Question 4 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

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	Model Solution	Scoring	
A	Find $g'(8)$. Give a reason for your answer.		
	$g'(x) = f(x)$	Considers $g'(x) = f(x)$	**Point 1 (P1)**
	$g'(8) = f(8) = 1$	Answer	**Point 2 (P2)**
	Scoring Notes for Part A		

- **P1** is earned for $g' = f$, $g'(x) = f(x)$, or $g'(8) = f(8)$ in part A.
- A response of $g'(8) = f(8) = 1$ earns both **P1** and **P2**.
- A response that does not earn **P1** can earn **P2** with an implied application of the Fundamental Theorem of Calculus, e.g., $g'(8) = 1$ or $f(8) = 1$.
- A response of $g'(8) = f(8) - f(6) = 1$ earns **P2** but not **P1**.

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B Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

The graph of g has a point of inflection where $g'' = f'$ changes sign, which is where $g' = f$ changes from decreasing to increasing or vice versa. The graph of g has points of inflection at $x = -3$ and $x = 6$ because f changes from decreasing to increasing there. The graph of g also has a point of inflection at $x = 3$ because f changes from increasing to decreasing there.	Answer	**Point 3 (P3)**
	Reason	**Point 4 (P4)**

Scoring Notes for Part B

- **P3** is earned only for an answer of $x = -3$, $x = 3$, and $x = 6$. If any other/additional values of x in $-6 < x < 12$ are declared to be points of inflection, the response does not earn either **P3** or **P4**. Consideration of $x = -6$ or of $x = 12$ does not impact scoring.
- To earn **P4**, a response must tie the reason to the given graph of f.
  - A response of "g has a point of inflection at $x = -3$, $x = 3$, and $x = 6$ because f changes from increasing to decreasing or decreasing to increasing there" earns both **P3** and **P4**.
  - A response of "g has a point of inflection at $x = -3$, $x = 3$, and $x = 6$ because the slope of f changes sign there" earns both **P3** and **P4**.
  - A response of "g has a point of inflection at $x = -3$, $x = 3$, and $x = 6$ because f attains relative extrema there" earns both **P3** and **P4**.
  - A response of "g has a point of inflection at $x = -3$, $x = 3$, and $x = 6$ because g changes concavity there" earns **P3** but not **P4**.
  - A response of "g has a point of inflection at $x = -3$, $x = 3$, and $x = 6$ because $g'' = f'$ changes sign there" earns **P3** but not **P4**.
  - A response that relies upon an ambiguous term such as "the function" or "the graph" does not earn **P4**.
- **Special case:** A response with two of the three correct x-values with correct reasoning, and no other/additional values of x declared to be points of inflection, earns **P4** but not **P3**.

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C Find $g(12)$ and $g(0)$. Label your answers.

$g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2} \cdot 6 \cdot 3 = 9$	$g(12)$	**Point 5 (P5)**
$g(0) = \int_6^0 f(x)\,dx = -\int_0^6 f(x)\,dx = -\frac{\pi}{2}3^2 = -\frac{9\pi}{2}$	$g(0)$	**Point 6 (P6)**

Scoring Notes for Part C

- Unlabeled values do not earn either **P5** or **P6**.
- **P5** is earned for a response of $g(12) = 9$, with or without supporting work.
- **P6** is earned for a response of $g(0) = -\frac{9\pi}{2}$, with or without supporting work.
  - Incorrect communication between the label "$g(0)$" and the answer will be treated as scratch work and will not impact scoring. For example, $g(0) = \int_0^6 f(x)\,dx = -\frac{9\pi}{2}$ earns **P6**.

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D Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

For $-6 \le x \le 12$, g attains a minimum either when $g'(x) = f(x) = 0$ or at an endpoint. $g'(x) = f(x) = 0$	Considers $g'(x) = 0$	**Point 7 (P7)**
$\Rightarrow \quad x = 0, \ x = 6$ $\begin{array}{c\|c} x & g(x) \\ \hline -6 & 0 \\ 0 & -\frac{9\pi}{2} \\ 6 & 0 \\ 12 & 9 \end{array}$	Justification	**Point 8 (P8)**
Therefore, on the closed interval $-6 \le x \le 12$, g attains an absolute minimum value at $x = 0$.	Answer	**Point 9 (P9)**

Scoring Notes for Part D

- **P7** is earned for considering $g'(x) = 0$ or $f(x) = 0$. **P7** is not earned by just presenting $x = 0$ and $x = 6$.
  - A response that discusses the sign of $g'(x)$ or $f(x)$ changing OR uses the phrase "critical points of g" also earns **P7**.
- To earn **P8** using a candidates test, a response must make a global argument by providing evaluations or reasoning for each of $g(-6)$, $g(0)$, $g(6)$, and $g(12)$ (and no other x-values).
- Alternate justification and answer:

  Because $g'(x) \le 0$ (or $f(x) \le 0$) for $-6 \le x < 0$ and $g'(x) \ge 0$ (or $f(x) \ge 0$) for $0 < x \le 12$, the absolute minimum of g occurs at $x = 0$.
- A response that presents a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P8** but is eligible for **P9** with the correct answer of $x = 0$.
- For **P8**, values of $g(0)$ and $g(12)$ can be imported from part C. A response can earn **P9** with an answer that is consistent with the imported values.

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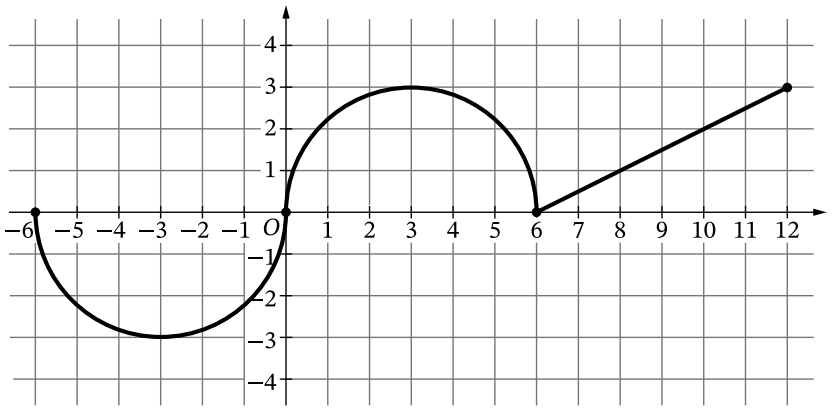
4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

(a) $g'(x) = f(x) \rightarrow g'(8) = f(8) = 1$

2: {P1: Considers $g'(x) = f(x)$; P2: Answer}

(b) g has points of inflections at $x = -3, 3,$ and 6 because $g'(x) = f(x)$ changes from decreasing to increasing or vice versa at these values of x.

2: {P3: Answer; P4: Reason}

(c) $g(12) = \int_6^{12} f(t)dt = \frac{1}{2}(6)(3) = 9$

$g(0) = \int_6^0 f(t)dt = -\int_0^6 f(t)dt = -\frac{1}{2}\pi(3)^2 = -\frac{9}{2}\pi$

2: {P5: $g(12)$; P6: $g(0)$}

(d) $g'(x) = f(x) = 0 \rightarrow x = 0,6$

x	$g(x) = \int_6^x f(t)dt$
-6	$-\int_{-6}^6 f(t)dt = -\left(-\frac{9}{2}\pi + \frac{9}{2}\pi\right) = 0$
0	$-\frac{9}{2}\pi$
6	0
12	9

g attains an absolute minimum at $x = 0$.

3: {P7: Considers $g'(x) = 0$; P8: Justification; P9: Answer}

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for *f* about $x = 1$.

C. The second-degree Taylor polynomial for *f* about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all *x* in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring	
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.		
	$\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y\frac{dy}{dx}$	Product rule	**Point 1 (P1)**
		Chain rule	**Point 2 (P2)**
	$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,y)=(1,\,-1)} = (3 - 1)(-1)^2 = 2$ $f''(1) = \left.\frac{d^2y}{dx^2}\right\|_{(x,y)=(1,\,-1)} = -(-1)^2 + (3 - 1)(2)(-1)(2) = -9$	$f''(1)$	**Point 3 (P3)**

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Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3-x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x, y, and $\frac{dy}{dx}$.
- A response of $-(-1)^2 + (3-1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.
- Alternate approach (using separation of variables):

  The particular solution for the differential equation that passes through the point $(1,-1)$ is $y = \frac{2}{x^2 - 6x + 3}$. Therefore, $\frac{dy}{dx} = \frac{-2(2x-6)}{\left(x^2 - 6x + 3\right)^2}$.

  This response has not yet earned **P1**, **P2**, or **P3**.
  - A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4\left(x^2 - 6x + 3\right)^2 + 4(2x-6)\left(x^2 - 6x + 3\right)(2x-6)}{\left(x^2 - 6x + 3\right)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
  - A response that correctly applies the quotient rule (or product rule), but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4\left(x^2 - 6x + 3\right)^2 + 4(2x-6)\left(x^2 - 6x + 3\right)}{\left(x^2 - 6x + 3\right)^4}$), earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
  - A response that does not correctly apply the quotient rule (or product rule), but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2\left(x^2 - 6x + 3\right)(2x-6)}$), does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

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B Write the second-degree Taylor polynomial for f about $x = 1$.

$f'(1) = 2$ and $f''(1) = -9$ $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$	Two terms	**Point 4 (P4)**
	Remaining term	**Point 5 (P5)**

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or "$+\cdots$" does not earn **P5**.
- A response of $-1 + 2(x-1) - \frac{9}{2}(x-1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x-1)$, but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

$\lvert f(1.1) - P_2(1.1)\rvert \le \frac{\max\limits_{1\le x\le 1.1}\lvert f'''(x)\rvert}{3!}\lvert 1.1-1\rvert^3 \le \frac{60}{6}(0.1)^3 = 0.01$	Form of error bound	**Point 6 (P6)**
	Analysis	**Point 7 (P7)**

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max\limits_{1\le x\le 1.1}\lvert f'''(x)\rvert}{3!}\lvert 1.1-1\rvert^3$ or $\frac{60}{6}(0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01, for example by communicating $\text{Error} \le 0.01$, $\text{Error Bound} = 0.01$, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

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D Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

$f(1.2) \approx f(1) + (1.2 - 1) \cdot \left.\frac{dy}{dx}\right\|_{(1, -1)}$ $= -1 + 0.2(2) = -0.6$ $f(1.4) \approx f(1.2) + (1.4 - 1.2) \cdot \left.\frac{dy}{dx}\right\|_{(1.2, -0.6)}$ $\approx -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.4704$ An approximation of $f(1.4)$ is -0.47.	First step of Euler's method **Point 8 (P8)** Answer with supporting work **Point 9 (P9)**

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.
  - Any subsequent error in simplification or rounding will not affect the scoring for **P8**.
- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

  - In the presence of a correct answer, a table does not need to be labeled in order to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled in order to earn **P8**.
- A response of $-0.6 + 0.2(3 - 1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47, $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.

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5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

(a) $\frac{d^2y}{dx^2} = \frac{d}{dx}[(3-x)y^2] = (-1)(y^2) + (3-x)\left(2y\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2} = -y^2 + (3-x)(2y)(3-x)(y^2) = -y^2 + 2y^3(3-x)^2$

$\left.\frac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + 2(-1)^3(3-1)^2 = -1 - 2 \cdot 4 = -9$

3: {P1: Product rule; P2: Chain rule; P3: $f''(1)$}

(b) $P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$

$= -1 + 2(x-1) + \frac{1}{2}(-9)(x-1)^2 = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$

2: {P4: Two terms; P5: Remainingh term}

(c) $|f(1.1) - P_2(1.1)| < \frac{\underbrace{\max}_{1 \le x \le 1.1} |f'''(x)|}{3!}(1.1-1)^3 = \frac{60}{6}(0.1)^3$

$= \frac{10}{1000} = \frac{1}{100}$

2: {P6: Form of error bound; P7: Analysis}

(d)

x	$y_{new} = y_{old} + \left.\frac{dy}{dx}\right\|_{(x,y_{old})}(\Delta x)$
1	-1
1.2	$y_{new} = -1 + [(3-1)(-1)^2]\left(\frac{1}{5}\right) = -\frac{6}{10}$
1.4	$y_{new} = -\frac{6}{10} + \left[(3-1.2)\left(-\frac{6}{10}\right)^2\right]\left(\frac{1}{5}\right) = -\frac{294}{625}$

2: {P8: First step of Euler's method; P9: Answer with supporting work}

6. The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

	Model Solution	Scoring	
A	Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.		
	$\left\lvert \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right\rvert = \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert$	Sets up ratio	**Point 1 (P1)**
	$\lim_{n\to\infty} \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert = \left\lvert \frac{x-4}{3} \right\rvert \lim_{n\to\infty} \frac{n+1}{n+2} = \left\lvert \frac{x-4}{3} \right\rvert$	Limit of ratio	**Point 2 (P2)**
	$\left\lvert \frac{x-4}{3} \right\rvert < 1$ when $-3 < x - 4 < 3$. Thus, the series converges when $1 < x < 7$.	Interior of interval of convergence	**Point 3 (P3)**
	When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3}{n+1}$, which converges by the alternating series test.	Considers both endpoints	**Point 4 (P4)**
	When $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{3}{n+1}$, which diverges by limit comparison to the harmonic series. Therefore, the series converges for $1 \le x < 7$.	Analysis and interval of convergence	**Point 5 (P5)**

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Scoring Notes for Part A

- **P1** is earned by presenting a correct ratio with or without absolute values. Once earned, **P1** is banked (i.e., subsequent errors in simplification or evaluation do not impact scoring for **P1**). **P2** is not earned if there are any errors in simplification or evaluation of the limit.
- **P1** is earned for ratios mathematically equivalent to any of the following:

$$\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-4)^{n+1}},\quad \frac{\dfrac{(x-4)^{n+2}}{(n+2)3^{n+1}}}{\dfrac{(x-4)^{n+1}}{(n+1)3^n}},\quad \frac{(x-4)^{n+1}}{(n+1)3^n}\cdot\frac{n3^{n-1}}{(x-4)^n},\quad \text{or}\quad \frac{\dfrac{(x-4)^{n+1}}{(n+1)3^n}}{\dfrac{(x-4)^n}{n3^{n-1}}}.$$

- **P1** is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{(x-4)^{n+1}}{(n+1)3^n}\cdot\frac{(n+2)3^{n+1}}{(x-4)^{n+2}},\quad \frac{\dfrac{(x-4)^{n+1}}{(n+1)3^n}}{\dfrac{(x-4)^{n+2}}{(n+2)3^{n+1}}},\quad \frac{(x-4)^n}{n3^{n-1}}\cdot\frac{(n+1)3^n}{(x-4)^{n+1}},\quad \text{or}\quad \frac{\dfrac{(x-4)^n}{n3^{n-1}}}{\dfrac{(x-4)^{n+1}}{(n+1)3^n}}.$$

  A response that presents any of these reciprocal ratios earns **P1** and is eligible for **P2**, **P3**, and **P4**, but not **P5**.
- A response that does not present a ratio is not eligible for **P2**.
- A response that does not use the absolute value of a ratio can earn both **P1** and **P2**.
- A response that does not use absolute value in computing the limit is eligible for **P3** if the expression $-1 < \frac{x-4}{3} < 1$ or an equivalent inequality is presented.
- A response that presents an incorrect limit of form $\frac{|x-4|}{b}$, where $b > 0$, is eligible for **P3**.

  P3 is then earned for correctly finding an interval with interior $(1, 7)$ or $(4-b, 4+b)$.

  Note: $|x-4| < b$ is not sufficient to earn **P3**.

  - $x - 4 < b$ can earn **P3** if this is resolved to $-b + 4 < x < b + 4$.
- **P4** is earned for considering both endpoints of the correct interval or both endpoints of an incorrect interval that has earned **P3**.
- To earn **P5**, a response must correctly analyze the series at $x = 1$ and $x = 7$, and present the correct interval of convergence. Naming of an appropriate test is sufficient for the analysis at each endpoint. In addition to the tests listed in the model solution, the direct comparison test to an appropriate series or the integral test may also be used.

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B Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

$f'(x) = \frac{(x-4)}{3} + \frac{(x-4)^2}{9} + \frac{(x-4)^3}{27} + \cdots + \frac{(x-4)^n}{3^n} + \cdots$	First three terms	**Point 6 (P6)**
	General term	**Point 7 (P7)**

Scoring Notes for Part B

- **P6** is earned by presenting the first three nonzero terms in a list or as part of a polynomial or series.
- **P7** is earned by identifying the correct general term (either individually or as part of a polynomial or series).

C The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

The Taylor series for f' is a geometric series with first term $\frac{x-4}{3}$ and common ratio $\frac{x-4}{3}$. $f'(x) = \frac{\frac{x-4}{3}}{1-\frac{x-4}{3}} = \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}$	Verification	**Point 8 (P8)**

Scoring Notes for Part C

- A response of $f'(x) = \frac{\frac{x-4}{3}}{1-\frac{x-4}{3}}$ is sufficient to earn **P8**.

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D It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

It follows from the work in part A that the interior of the interval of convergence of the Taylor series for f' is $1 < x < 7$. Therefore, $x = 8$ would be outside the interval of convergence of f', and the Taylor series for f' would not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.	Answer with reason **Point 9 (P9)**

Scoring Notes for Part D

- A response of "no, $x = 8$ is outside the interval of convergence" is sufficient to earn **P9**.
- **P9** can be earned with a response consistent with an incorrect interval of convergence imported from part A.
- Alternate solutions:
  - Because the series for $f'(x)$ is geometric, this converges to $f(x)$ for all values of x such that the common ratio $\frac{x-4}{3}$ is between -1 and 1.

    $-1 < \frac{x-4}{3} < 1 \Rightarrow -3 < x-4 < 3 \Rightarrow 1 < x < 7$
  - Because $x = 8$ is outside the interval $1 < x < 7$, the series for f' does not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.
  - For $x = 8$, the series is given by $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$, which is a geometric series with $r = \frac{4}{3} > 1$. Therefore, the series diverges for $x = 8$.

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6. The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2\cdot 3} + \frac{(x-4)^3}{3\cdot 3^2} + \frac{(x-4)^4}{4\cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)}{(n+2)} \cdot \frac{x-4}{3}\right| = \left|\frac{x-4}{3}\right| < 1$

$|x-4| < 3 \to 1 < x < 7$

$x = 1 \to \sum_{n=1}^{\infty} \frac{(1-4)^{n+1}}{(n+1)3^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{(n+1)3^n} = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$

which converges by the alternating series test at $x = 1$

$x = 7 \to \sum_{n=1}^{\infty} \frac{(7-4)^{n+1}}{(n+1)3^n} = \sum_{n=1}^{\infty} \frac{(3)^{n+1}}{(n+1)3^n} = 3\sum_{n=1}^{\infty} \frac{1}{(n+1)}$

which diverges at $x = 7$ by the limit comparison test with the harmonic series .

The interval of convergence is $1 \le x < 7$.

5: { P1: Sets up ratio; P2: Limit of ratio; P3: Interior of interval of convergence; P4: Considers both endpoints; P5: Analysis and interval of convergence }

(b) $\frac{2(x-4)}{2\cdot 3} + \frac{3(x-4)^2}{3\cdot 3^2} + \frac{4(x-4)^3}{4\cdot 3^3} + \cdots + \frac{(n+1)(x-4)^n}{(n+1)\cdot 3^n} + \cdots$

$\frac{(x-4)}{3} + \frac{(x-4)^2}{3^2} + \frac{(x-4)^3}{3^3} + \cdots + \frac{(x-4)^n}{3^n} + \cdots$

2: { P6: First three terms; P7: General term. }

(c) $f'(x) = \sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x-4}{3}\right)^n$ which converges to

$S = \frac{a}{1-r} \to \frac{\frac{(x-4)}{3}}{1-\frac{(x-4)}{3}} = \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}$

P8: Verification

(d) No the Taylor series for f' does not converge to $\frac{x-4}{7-x}$ because $x = 8$ is not in the interval $-1 \le x < 7$.

P9: answer with reason